

BOOK REVIEW

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Duality and perturbation methods in critical point theory, by Nassif Ghoussoub.
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There are many problems in mathematics and physics, for instance in geometry and mechanics, whose solutions are extrema of some kind of energy. Geodesics on Riemannian manifolds, minimal surfaces, conformal metrics with prescribed curvature properties, harmonic maps, as well as solutions to Hamiltonian systems or semilinear elliptic partial differential equations are critical points of a suitable energy (or action) functional. The least action principle of Euler and Maupertuis even expresses the belief that every law of nature can be formulated as an extremum principle.

The classical calculus of variations whose origins can be traced back to Fermat, Newton, and the Bernoulli brothers deals with the problem of studying properties of an extremum—once the appropriate “energy” has been found. The Euler-Lagrange equations which are necessarily satisfied by any extremum have been investigated by many mathematicians ever since their discovery in the middle of the eighteenth century. However, the existence of, say, a minimum was taken for granted if the energy was bounded below. It was not before the second half of the nineteenth century that existence was considered as a problem. In 1870 Weierstraß gave an example of a variational problem with nonnegative energy which did not have a minimum solution. The breakthrough concerning the question of existence came around 1900 with the work of Hilbert, who justified the so-called Dirichlet principle, which claims the existence of a harmonic map with given boundary data and which was used frequently during the nineteenth century, for instance, by Gauß and Riemann.

After the justification of the Dirichlet principle the calculus of variations was developed systematically hand in hand with functional analysis. Two geometric problems in particular presented a big challenge (they still do) and led to remarkable theories. These problems are the existence (and number) of closed geodesics on a compact Riemannian manifold and the existence (and number and topological type) of minimal surfaces bounded by a given closed curve. In 1929 Lusternik and Schnirelmann proved the existence of three distinct closed geodesics on a compact surface of genus zero. In 1930–31 Douglas and Rado solved the Plateau problem by finding a minimal surface which minimizes the energy. Around 1940 Morse, Tompkins, and Shiffman discovered unstable “minimal” surfaces. These as well as the closed geodesics are not minimizers of the energy but critical points with Morse

index different from zero. Clearly unstable critical points are more difficult to find. A skier starting at the top of a mountain will most likely end up in a valley and has to be quite skillful if he wants to stop at a saddle point. Taking all critical points of the energy functional into account leads to the “calculus of variations in the large” or “global analysis”. This is an important step that greatly enriched the subject. Many functionals do not have a minimizer. For example, the action integral which one associates to a first-order Hamiltonian system and whose critical points are the periodic solutions is unbounded from below and above. Moreover, even if a minimizer exists, one is often interested in finding all solutions of the corresponding Euler-Lagrange equations.

The theories of Lusternik and Schnirelmann (LS-theory, for short) and of Morse provide a deep understanding of the relation between the topology of the space of functions on which the energy is defined and the number and type of critical points. Both approaches have been extended and applied to numerous problems from mathematical physics to differential geometry and topology. Recent versions of LS-theory and Morse theory are now capable of dealing with variational principles which—a few years ago—seemed to be too degenerate to be of any use. The work of Rabinowitz and Floer, for instance, on Hamiltonian systems and symplectic geometry is a model example for this development. Although both theories are closely related, they developed somewhat differently. The book by Ghoussoub is mainly concerned with LS-theory, so we concentrate on that and leave aside the fascinating developments of Morse theory associated with the work of Thom, Smale, Milnor, Witten, and Floer.

The basic idea for finding a minimum is very simple. We let E denote the energy functional defined on a Banach space or Banach manifold V and assume E is of class \mathcal{C}^1 , as always in the book and this review. (This assumption is much too strong for most applications of the direct methods, but these are not the concern of the book.) If E is bounded below, we take any minimizing sequence (u_n) , that is, $E(u_n) \rightarrow \inf E$ as $n \rightarrow \infty$. The main problem is to show that the minimizing sequence has an accumulation point which will then be a minimizer. The existence of an unstable critical point leads to a similar compactness problem. For example, think of the graph of E as a landscape, E denoting the height. Suppose the point $u_0 \in V$ lies in a valley surrounded by a mountain range of minimal height $\alpha > E(u_0)$, that is, $E(u) \geq \alpha$ for every u in the boundary of a bounded neighborhood U of u_0 . Suppose, moreover, that there is a point u_1 outside of U of height $E(u_1) < \alpha$. In order to go from u_0 to u_1 one has to pass the boundary of U , if possible on a path of minimal height. Thus, if P denotes the set of all paths $p: [0, 1] \rightarrow V$ with $p(0) = u_0$, $p(1) = u_1$, one tries to find a mountain pass $p_M \in P$ such that

$$\text{height of } p_M = \max_{0 \leq t \leq 1} E(p_M(t)) = \inf_{p \in P} \max_{0 \leq t \leq 1} E(p(t)) =: c_M.$$

The real number c_M is called a minimax value of E . It is possible to construct a sequence (u_n) with $E(u_n) \rightarrow c_M$ and $E'(u_n) \rightarrow 0$. If an accumulation point u_M exists, it will be an unstable critical point, a saddle point, with Morse index one (if u_M is nondegenerate) and critical value c_M .

The Palais-Smale condition requires that every sequence (u_n) with $E(u_n)$ bounded and $E'(u_n) \rightarrow 0$ has a convergent subsequence. It was introduced around 1963 by Palais and Smale, who developed Morse theory and LS-theory on Hilbert and Banach manifolds. During the 1960s the Palais-Smale condition was received with

some reservation. Except for one-dimensional problems like the closed geodesic problem, it did not seem to hold for most other interesting problems from differential geometry. However, the Palais-Smale condition is a good condition in a dual sense. If it is satisfied for the energy function E , then one only needs to find values $c \in \mathbb{R}$ where the sublevel sets $E^c = \{u \in V : E(u) \leq c\}$ change topology, as in the case of the infimum or the mountain pass level. A corresponding $(PS)_c$ -sequence, that is, a sequence (u_n) with $E(u_n) \rightarrow c$ and $E'(u_n) \rightarrow 0$, will then have a critical point as accumulation point. If, on the other hand, the Palais-Smale condition is not satisfied, then it points the way to further research. Namely, one should study $(PS)_c$ -sequences that do not have an accumulation point and understand the way this can happen.

Despite the initial scepticism, the Palais-Smale condition has found numerous applications. Even if it does not hold in general, often weaker versions still do. For example, it may hold below a certain level S , which means that all $(PS)_c$ -sequences with $c < S$ have a convergent subsequence. This happens for instance in the case of the Yamabe problem, where one wants to find a conformal metric with constant scalar curvature on a Riemannian manifold. Another approach applies to certain limit cases (again the Yamabe problem is an example) where one can find perturbations E_λ of the given functional $E = E_0$ so that the Palais-Smale condition holds for E_λ if $\lambda > 0$ but not for E_0 . Then one can try to find critical points u_λ of E_λ and show that u_λ converges as $\lambda \rightarrow 0$. This idea has been applied by Trudinger, Aubin, and Schoen in their solution of the Yamabe problem (1968–1984). It has also been used by Sacks and Uhlenbeck (1981) in order to find harmonic maps from a closed surface to a Riemannian manifold. Investigating in what sense the u_λ converge, they discovered a very interesting geometric phenomenon, the “separation of spheres” (the image of the surface under u_λ forms bubbles which split off as λ approaches zero). Similar phenomena can also be observed for other functionals when one studies $(PS)_c$ -sequences that do not have a convergent subsequence.

One of the main goals of Ghoussoub’s book is the construction of $(PS)_c$ -sequences which have special properties. These additional properties can then be used to find a convergent subsequence even if an arbitrary $(PS)_c$ -sequence need not have an accumulation point. Two methods are developed in detail, as the title suggests. The duality method consists of finding a second “dual” description of a minimax value. For example, recall that the mountain pass value is given by $c_M = \inf \max E(p(t))$ where the infimum is over certain paths p and the maximum is over $t \in [0, 1]$. A dual description interchanges \inf and \max : $c_M = \sup \inf E(u)$ where the supremum is over a certain class of subsets F of V and the infimum over $u \in F$. Such a dual description allows one to localize the associated $(PS)_c$ -sequences, that is, to obtain information on their position in V . The perturbation method consists of passing from the given functional $E = E_0$ to a family of perturbations E_λ which have better properties. In particular, one would like the Palais-Smale condition to hold for E_λ if $\lambda > 0$ and that the critical points of the perturbed functionals are nondegenerate. Then a minimax value c_λ yields a critical point u_λ of E_λ whose Morse index can be controlled. Both the localization of a $(PS)_c$ -sequence as well as information on Morse indices may help to establish a convergent subsequence. The two techniques can also be combined. An additional benefit of the localization method is that it allows one to weaken the assumptions of various abstract critical point theorems. In the situation of the mountain pass theorem, for instance, one

can treat the extreme case where the range of mountains around the point u_0 is bounded below only by the height of u_0 , not by a number $\alpha > E(u_0)$.

The book is carefully written with only a small number of misprints and mistakes which do not puzzle the experienced reader. It has a certain technical flavour and emphasizes the abstract LS-theory with the applications serving as illustrations. Although the abstract theory can in principle be read by anyone with a working knowledge of functional analysis/analysis on Banach manifolds, none of the applications are on this level. The first example on page 8 is already a nonhomogeneous elliptic equation with critical exponent where the Palais-Smale condition does not hold in general. Also, most of the other examples and applications are chosen so that the more elementary techniques of critical point theory do not apply directly. Not treated are strongly indefinite functionals where the Morse indices of the critical points are infinite, hence there are no applications to Hamiltonian systems.

For a reader who does not know the standard theory I suggest starting with the recent monographs by Rabinowitz, Mawhin-Willem, and Struwe. The book by Struwe also contains a chapter on problems where the Palais-Smale condition does not hold. I recommend Ghoussoub's book to anyone working on variational problems, in particular on LS-theory. The reader will find a systematic presentation and detailed exposition of useful techniques.

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