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T. Bartsch
Mathematisches Institut, Universität Heidelberg
E-mail address: bartsch@harmless.mathi.uni-heidelberg.de

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Nonlinear parabolic and elliptic equations, C. V. Pao. Plenum Press, New York and London, 1992, xv+777 pp., $125.00. ISBN 0-306-44343-0

The goal of this book, as stated by the author, is to present a systematic treatment of the basic mathematical theory for certain classes of nonlinear parabolic and elliptic partial differential equations using the method of upper and lower solutions.

A considerable amount of study has been devoted to establish the existence of solutions for elliptic and parabolic problems and their basic properties, provided upper and lower solutions of such problems exist. Such techniques have been used for many years. In 1915 in one of the earliest such applications Perron [1] used solutions of differential inequalities to establish the existence of a solution of the initial-value problem for first-order ordinary differential equations. Much of the work on elliptic and parabolic problems has its basis in the fundamental paper of Nagumo [2] as carried further by Ako [3]. Keller [4] and Amann [5] constructed solutions between upper and lower solutions of elliptic problems using a monotone iteration scheme which was possible because the nonlinear reaction term was assumed to be gradient independent and because of certain one-sided Lipschitz continuity assumptions upon the nonlinear terms. Schmitt [6] has a good survey of such results. Sattinger [7] extended Amann's results to parabolic initial-boundary value problems using a similar monotone iteration scheme. This work was subsequently extended to various kinds of problems for both elliptic and parabolic equations by Pao [8, 9] and Puel [10] using either monotone iteration techniques or the theory of monotone operators. In these latter papers a minimal solution is produced by starting the iteration scheme with the given lower solution and the maximal solution by commencing the scheme with the upper solution. This procedure has certain computational advantages for the class of admissible reactive nonlinearities and can be considered constructive.

This monograph is an outgrowth of the author's research in this area, tracing back to his earlier papers mentioned above. The dominant theme is to assume the existence of appropriate pairs of upper and lower solutions for the nonlinear problem under consideration, develop a monotone iteration process, and determine if classical solutions exist. There is no doubt that this is a constructive, powerful method that is systematically and carefully used by Pao to attack one class of problems after another. But a monograph costing $125 and consisting of 756 pages of exposition seems a bit excessive.

The monograph consists of twelve chapters and an extensive list of references (432 to be exact). To give a more detailed flavor of the monograph, each chapter will now be summarized.

1. **Reaction diffusion equations.** The method of upper and lower solutions and the method of monotone iterations are introduced for time-dependent and steady-state RD equations. A formal derivation of a number of mathematical models resulting in RD equations is given from various fields of applications.

2. **Parabolic boundary value problems.** After a review of the fundamental theory of linear parabolic problems, the standard positivity lemma based on the maximum principle is proven. The use of upper and lower solutions as initial iterates leads to two monotone sequences, each of which converges to a unique solution of an integral equation. Under weak smoothness assumptions on the reaction term, these limits are solutions of the given parabolic problem for the three standard boundary conditions. Extensions to nonlocal integroparabolic and functional-parabolic problems are discussed.

3. **Elliptic boundary value problems.** Monotone sequences for elliptic boundary value problems are developed, and conditions are given to guarantee that the limits of such sequences are classical solutions. Uniqueness and multiplicity results are proven. These ideas are extended to certain classes of nonlocal problems, and several applications of the theory are given.

4. **Equations with nonlinear boundary conditions.** A number of the earlier results are extended to problems with nonlinear boundary conditions by the method of upper and lower solutions.

5. **Stability analysis.** If suitable pairs of upper and lower solutions can be constructed for certain classes of parabolic boundary value problems, then the global existence of solutions is guaranteed and the asymptotic behavior can sometimes be determined. When the initial function in the parabolic problem is an upper solution or a lower solution of the corresponding elliptic problem, the solution of the parabolic problem is monotone in t and converges to a steady-state solution. Sufficient conditions for a given steady-state solution to be stable are given.

6. **Blowing-up behavior of solutions.** For initial-boundary value problems for nonlinear parabolic equations, solutions may exist locally and hence on a maximal time interval. If this maximal time is finite and if some norm of the solution becomes unbounded as this maximal time is approached, the solution is said to blow up in finite time. If an initial-boundary value problem has a nonnegative lower solution that becomes unbounded in finite time, then any solution of such a problem blows up in finite time and one has an upper bound on the blow-up time. This observation leads to a theorem that was first observed by Kaplan. A second method for determining blowup was developed in the seventies by Levine and Payne [11]. This method is based on the concavity of a

certain functional. Both methods are developed in this chapter. The question of where in the spatial domain blowup can occur is only touched upon, and only single-point blowup as developed by Friedman and McLeod [12] is presented. Quenching problems are mentioned.

7. **Parabolic and elliptic equations in unbounded domains.** In this chapter, the author extends to unbounded domains the method of upper and lower solutions for parabolic and elliptic equations in bounded domains. This extension is straightforward. For the radially symmetric elliptic problem, a construction of infinitely many solutions is given.

8. **Coupled systems of reaction diffusion equations.** The method of upper and lower solutions is extended to coupled systems of parabolic and elliptic equations. For this extension the reaction nonlinearity is assumed to be quasimonotone. Without assuming quasimontonity, existence is proven by a contraction argument and by introducing generalized upper and lower solutions as was first done by Ako.

9. **Systems with nonlinear boundary conditions.** Coupled systems of parabolic and elliptic equations are considered. The coupling can be through nonlinear boundary conditions. Under the assumption of quasimonotoneity, the method of upper and lower solutions is extended to this situation. When quaismonotoneity is dropped, existence can be proved using degree-theoretic arguments, provided generalized upper and lower solutions exist.

10. **Stability and asymptotic behavior of solutions.** A stability analysis for boundary value problems for coupled systems is developed assuming quasimontoneity of the reaction terms. For a given steady-state solution, sufficient conditions are obtained to ensure its asymptotic stability.

11. **Asymptotic limit and blowing up behavior of solutions.** For a certain class of Neumann boundary value problems with a special reaction term, the steady-state problem possesses an infinite number of nonisolated constant solutions. Convergence of the time-dependent solution to one of these constant steady states is proven, and its exact value is determined. For two coupled parabolic equations, finite time blowup is shown to occur in a few special cases.

12. **Applications of coupled systems to model problems.** The theory developed in the previous chapters for systems of parabolic and elliptic equations is applied to a number of models taken fron several areas of applied science and engineering. These models include the Belousov-Zhabotinskii system for catalyzed oxidation, the classical Volterra-Lotka ecological model, the FitzHugh-Nagumo equations, and the Kermack-McKendrick model.

Focusing primarily on problems that can be handled by monotone iterative methods does limit the scope of the book. The advantage of such methods is that they should be amenable to constructive numerical development. Omission of any such development in the book weakens this argument, unfortunately. The disadvantage is that to use such methods requires rather severe assumptions on monotoneity of the nonlinear terms. If one is simply trying to develop a fundamental theory for semilinear and quasilinear problems, then other methods, some of which are based on generalized upper and lower solution techniques (see Carl [13], for example), are needed.

There are some weaknesses in the exposition. The proof on page 122 of the existence of at least three solutions of the nonlinear eigenvalue problem for the elliptic boundary value problem with Dirichlet boundary conditions assuming the existence of appropriately ordered upper and lower solutions is reduced to essentially saying that by using Leray-Schauder degree theory and the fixed-point index, Amann [14] has proven the existence of a third solution. Absolutely no insight is given as to what is making the result true, and no mention is made of the degree-theoretic results being used. There are several other times when topological tools are called upon without motivation.

A great deal of work was required to develop this large set of lecture notes and turn it into a carefully written massive monograph on reaction-diffusion equations. It can usefully serve as a source for anyone interested in monotone iterative methods.

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JERROLD BEBERNES
UNIVERSITY OF COLORADO
E-mail address: `bebernes@newton.colorado.edu`