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Model theory, by Wilfrid Hodges. Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993, xiii + 772 pp., $99.95. ISBN 0-521-30442-3

Mathematical logic began as the general study of mathematical reasoning. Several specializations have developed: recursion theory studies abstract computation; set theory studies the foundations of mathematics as formalized in Zermelo-Fraenkel set theory; proof theory studies systems of formal proof; model theory, says Hodges, is the study of the construction and classification of structures within specified classes of structures. Despite this somewhat jargon-laden definition, model theory is the part of logic that over the last half century has developed the deepest contacts with core mathematics. I would modify Hodges's definition to emphasize a central concern of model theory: the discussion of classes of classes of structures. In this review we consider several developments, most covered by Hodges, to clarify the distinction between studying a single structure, a class of structures (a theory), or a class of theories. We will relate these model-theoretic concepts to specific developments in the theory of fields.

A signature L is a collection of relation and function symbols. A structure for that signature (L-structure) is a set with an interpretation for each of those symbols. Consider a signature L with symbols: $<$, $+$, $\cdot$. The ordered field of real numbers becomes an L-structure when these symbols are given their natural interpretations. A language specifies certain L-sentences built up from the symbols in a signature L. A number of languages can be built up based on the same signature. The first-order language is the least set of formulas containing the atomic L-formulas (e.g., $x < y$) and closed under the Boolean operations and quantification over individuals. Formulas in which each variable is bound by a quantifier are called L-sentences. More expressive languages allow closure under infinite conjunctions ($L_{\kappa,\omega}$), quantifiers over subsets (second-order logic), etc. An L-structure M is a model of an L-sentence if the sentence is true in M. Thus the class of real closed fields is axiomatizable in the first-order language associated with L, while the reals are axiomatizable in the second-order language associated with L.

Although we focus most of our attention on the first-order case, results about individual structures often involve more expressive languages. A specimen result of this sort asserts that an element m of a countable structure M is fixed by every automorphism of M that fixes $X \subseteq M$ if and only if m is $L_{\omega_1,\omega}$-*definable with parameters* from X (i.e., there is a formula $\phi(v, \bar{w})$ of $L_{\omega_1,\omega}$ and elements $\bar{x}$ from X such that m is the only solution of $\phi(v, \bar{x})$).

In a given context a signature L and a language (say, first order) are fixed. A theory T is a collection of L-sentences. If all models of T satisfy exactly the same L-sentences (are elementarily equivalent), T is said to be *complete*. We consider two examples of the utility of extending our focus from a single structure to the class of all structures elementarily equivalent to M. There are many such models: the Löwenheim-Skolem theorem asserts: if a first-order theory has an infinite model, it has a model in each infinite cardinality.

Tarski [17] characterized the first-order definable subsets of the ordered field of real numbers as the Boolean algebra generated by intervals. This is a *quantifier elimination* result, as (when suitably generalized to n-space) it shows that every first-order definable relation is in fact definable without quantifiers (by a Boolean combination of equalities and inequalities). In fact, this elimination can be proved for all real closed fields and leads to the conclusion that the theory of real closed fields is complete. A corollary is the (only known?) proof that every associative division ring over an arbitrary real closed field has dimension at most 8. This theorem is proved by analytical means for the reals and transferred to fields where such analysis is impossible.

The compactness theorem asserts that if every finite subset of a set Σ of sentences has a model, then so does Σ. This theorem allows one to realize "ideal" elements in closely related structures and provides us with a concrete notion of the meaning of a generic point in algebraic geometry. To understand this, we need the notion of a *type*. Let M be a structure, and let $A \subseteq M$ and $m \in M$. The type realized by m over A in M is the collection of formulas with parameters from A satisfied by m in M. For every set A contained in a model M and every n, let $S_n(A)$ denote the collection of types over A (the *nth Stone space* of A). If A is embedded in a sufficiently nice "universal domain", $\mathcal{M}$, all realizations in $\mathcal{M}$ of a type $p \in S_n(A)$ are conjugate by automorphisms of $\mathcal{M}$ fixing A.

Now, let k be an algebraically closed field. A variety V can be viewed syntactically as a set definable by a conjunction of equations. A type $p \in S_n(k)$ is *generic for* V if the formula defining V is in p and every formula in p has dimension at least that of V either in the sense of algebraic geometry or, equivalently, in the sense of Morley (see below). A generic point for a variety V defined over the algebraically closed field k is a realization in an elementary extension k' of k of a generic type for V. That is, a generic point is a point like any other—but in a suitably chosen elementary extension of the ground field.

Work on particular theories naturally tends to focus on theories with mathematical significance. Some of the more important recent work includes Wilkie's [18] proof of the model completeness of the real field with exponentiation and the work of Macintyre, van den Dries and Denef on the p-adics [12, 9]. By replacing varieties with all definable sets, van den Dries and Denef are able to substitute an argument using induction on the logical complexity of formulas for one using resolution of singularities.

The core of model theory investigates the common properties of members of a class of theories. A natural starting place is to sort classes of theories according to the syntactic nature of the sentences defining the theory. Ideally, one proves equivalences between semantic and syntactic characterizations of a class. Thus, the Chang-Łoś-Suszkó theorem says that a theory T is axiomatized by universal-existential sentences if and only if the class of models of T is closed under unions of chains.

The study of classes of theories defined by more structural rather than syntactic properties leads to deeper model-theoretic results. By the Löwenheim-Skolem theorem it is impossible for a first-order theory to specify models more precisely than up to cardinality. The algebraically closed fields of any fixed characteristic provide an example of a first-order theory that is categorical (has only one model up to isomorphism) in each uncountable cardinality. Łoś [11] conjectured that this is a general phenomenon: a theory categorical in one uncountable cardinal κ is categorical in all uncountable cardinalities.

Morley [14] made the pivotal steps toward a sophisticated analysis of complete first-order theories in his positive solution of the Łoś conjecture. Among the numerous crucial ideas he introduced in this paper, we consider his analysis of Stone spaces. If M is an uncountable model of an $\aleph_1$-categorical theory T, the Morley rank of $p \in S_1(M)$ is just the Cantor-Bendixson rank of p as a member of this Stone space. Morley uses this rank to provide an inductive procedure for analyzing the complexity of models. In current terminology, a theory T is called κ-stable if $|S_1(M)| = \kappa$ for every model of T with cardinality κ. Every type has a Morley rank just if T is ω-stable. Morley showed that every $\aleph_1$-categorical theory is ω-stable.

In retrospect, Morley's paper demonstrates the advantage of investigating complete first-order theories that satisfy additional restrictive conditions (e.g., ω-stability). But this lesson was not learned immediately. The attempt to consider such ideas as two cardinal models, saturated and special models, and omitting types for arbitrary theories led to a set-theoretic morass. Shelah seized on the notion of classifying theories by properties of associated Stone Spaces to avoid the set-theoretic independence questions that arose when these constructions were considered for arbitrary theories. He introduced a classification of all first-order theories in terms of the cardinals κ in which T is stable. In particular, T is stable if T is stable in some infinite cardinal. He generalized from the Łoś conjecture to the more general problem of computing $I(T, \lambda)$, the number of models of a first-order theory T that have cardinality λ. In a long project Shelah used his classification to determine the spectrum functions $I(T, \lambda)$ of first-order theories. The spirit of this program is more fully explained in [15, 2, 5]. Among the surprising but decisive side-effects of this project was the discovery that methods invented to discuss models of large cardinality or in signatures with large cardinality had profound consequences for countable models of countable theories. (This observation complements Shelah's remark [16] reported on page 339 of Hodges's book.)

One key aspect of the Shelah program is the development in the general context of stable theories of a notion of dependence similar to algebraic dependence in vector spaces or fields. Hodges chooses to expound this notion most fully in the much more restrictive context of almost strongly minimal theories. A definable set D is strongly minimal if every definable subset of D is (uniformly) finite or cofinite. There is a natural combinatorial geometry on the algebraically closed subsets of D. Around 1970 Baldwin and Lachlan [3] proved that an $\aleph_1$-categorical theory was controlled by a strongly minimal set; this control is most direct in the case of an almost strongly minimal theory [1]. At about the same time, Zil′ber began to develop the importance of the geometric structure of the strongly minimal set in determining the properties of the theory. He isolated the crucial notion of a modular (essentially, the lattice of closed sets is modular) strongly minimal set and characterized strongly minimal sets with locally finite modular geometries. This characterization is part of his proof that no first-order theory with only infinite models that is categorical in every infinite power can be finitely axiomatizable [19]. Zil′ber conjectured that every nonmodular strongly minimal set is "biinterpretable" with an algebraically closed field. This conjecture reduces the study of arbitrary interesting uncountably categorical theories to algebraic geometry. We will return to this situation after addressing the book at hand.

The last book entitled *Model theory* was written by Chang and Keisler [4] in 1973. Two major topics in Chang-Keisler are largely omitted in Hodges: ultraproducts and the model theory of set theory. Each of these areas reflect a kind of interaction with set theory which is no longer central to model theory. Both focus on models of a particular theory.

In the late 1960s the study of infinitary logics was a flourishing field. Chang and Keisler deliberately excluded it from their text; Keisler wrote another book on that subject [10]. Hodges views model theory as a whole with first-order logic as only one case. Thus, the compactness theorem is delayed until page 264, emphasizing that, for example, the analysis of automorphism groups of structures is not dependent on the restriction to the first-order situation.

Chang and Keisler's book was carefully organized as a graduate text. In contrast, Hodges's book should be viewed more as an encyclopedia than a text. It surveys most of contemporary model theory: not only the classical results from before 1960, but most of the exciting developments of the last thirty years are covered. Listing such topics as the small index property, ω-categoricity, back and forth techniques, quantifier elimination, stable groups, Horn theories, Ehrenhfeucht-Mostowski models, and cohomology of expansions only scratches the surface. Morley's theorem is one of a number of subjects discussed at the end of Chang-Keisler. The study of classes of theories, defined mainly by stability considerations, plays a major role in Hodges's book. This approach enables one to distinguish properly model-theoretic notions which are reasonably robust modulo set theory from problems that are inherently set-theoretical. Hodges presents Shelah's proof that an unstable theory has the maximal possible number of models in each regular uncountable cardinal. The restriction to regular cardinals exemplifies this distinction. The proof that there are 2^λ models of power λ for arbitrary λ [16] involves both model-theoretic ideas and combinatorial properties of λ. For more exotic λ, both more set theory and more model theory are required, but the regular cardinal case contains the essential ideas.

The book is valuable to all mathematicians with some model-theoretic background because Hodges's organization and informal style bring out unexpected connections and unique insights on many topics. I disagree with one of these remarks. Hodges objects to the distinction between $\vdash$ for syntactic consequence and $\models$ for semantic consequence and asserts that most model theorists use $\vdash$ for semantic consequence. A cursory survey has found no one else has adopted this convention.

An example of Hodges's novel organization is the proof early in the book of the result of Zil′ber and Hrushovski that the existence of certain simple geometric configurations in a structure imply that an infinite group can be interpreted in the structure. The idea of this proof can be expressed at several levels. Hodges takes the restrictive assumption that the structure is almost strongly minimal to avoid certain complications and states the result in terms of type definability. By strengthening the hypotheses and weakening the conclusion (from the most general result), he focuses on the crux of the proof.

Some work that appeared too late for inclusion in Hodges's book completes our story. Hrushovski used the result described in the last paragraph to derive abstract model-theoretic results from concrete algebraic facts. But he also [6] refuted Zil′ber's conjecture limiting the strongly minimal sets to those derived from standard algebra.

Model theory has provided a textbook example of the benefits gained from abstracting from a situation, working our the consequences of the abstraction, and returning again to the specific situation. Morley's theorem seizes on a salient aspect of algebraic geometry: the categoricity of the field of complex numbers. The generalizations to all first-order theories categorical in an uncountable power and then to classes of theories defined by stability considerations produced a powerful machinery free from the specifics of the situation. The development of stability theory allowed a uniform treatment of such disparate structures as differentially closed fields and separably closed fields. To regain the Zil′ber conjecture, Hrushovski and Zil′ber developed the theory of Zariski structures [8]. This notion of a restricted type of ω-stable theory gives a model-theoretic axiomatization of exactly the dimension theory that arises in algebraic geometry. It specializes the generality of general stability theory while giving a structural, rather than syntactic, analog for the notion of variety. The fruitfulness of this approach has been seen in the solution of both model-theoretic and algebraic problems: calculation of the number of countable models of the theory of differentially closed fields of characteristic 0 [7] and Hrushovski's work on the Mordell-Lang conjecture in both characteristics 0 and p. An essential ingredient in the characteristic p case is Messmer's characterization [13] of definable groups over separably closed fields: a strictly stable structure.

Hodges's book does not carry the story quite this far. However, it expounds many of the crucial ideas used in this analysis and provides a beautiful panorama of most of the important results in model theory.

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