BOOK REVIEW

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The Riemann zeta function, by A. A. Karatsuba and S. M. Voronin. de Gruyter, Berlin and Hawthorne, New York, 1991, xii + 396 pp., $112.00. ISBN 3-11-013170-6

The Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}(s) > 1$. It has a meromorphic continuation to $\mathbb{C}$ with the only pole being at $s = 1$. Moreover it satisfies a functional equation relating s and $1-s$. Its relevance to prime numbers lies in the relation $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$. The central problem concerning $\zeta(s)$ is *Riemann's Hypothesis* (RH) that the zeros of $\zeta(s)$ in $0 < \mathrm{Re}(s) < 1$ are all on $\mathrm{Re}(s) = 1/2$. For many applications the following weaker hypotheses suffice:

Lindelof Hypothesis: (LH). For any $\varepsilon > 0$ there is C_ε such that $|\zeta(\frac{1}{2} + it)| \leq C_\varepsilon (1 + |t|)^\varepsilon$.

Density Hypothesis: (DH). If $N(\sigma, T) = |\{\rho : \zeta(\rho) = 0, |\mathrm{Im}(\rho)| \leq T, \mathrm{Re}(\rho) \geq \sigma\}|$, then for $\varepsilon > 0$, $\sigma \geq \frac{1}{2}$, there is $C_{\varepsilon,\sigma}$ such that $N(\sigma, T) \leq C_{\varepsilon,\sigma} T^{2-2\sigma+\varepsilon}$.

We remark that $N(\frac{1}{2}, T) \sim \frac{T}{2\pi} \log T$, as was noted by Riemann and RH $\Rightarrow$ LH $\Rightarrow$ DH.

The book under review centers around these problems and their variants. To quote the inside cover of the 1951 edition of Titchmarsh's Riemann Zeta Function Titchmarsh [T1]: "... but some of the main problems in the theory, such as the Riemann Hypothesis, are still unsolved, so that a definitive work on the subject is still not possible." That statement is as valid today as it was in 1951. Nevertheless, deep progress has been made in understanding $\zeta(s)$, and important general techniques have been developed for this purpose. Titchmarsh [T1] gives a good account of this work prior to 1951. The new edition Titchmarsh [T2] with the revision by Heath-Brown gives a good update of the developments between 1951 and 1986, but the details of the revision are necessarily sketchy, being in the form of notes at the end of the chapters.

In this book the authors have chosen to expose in detail certain aspects of recent developments. Very strong results in the direction of LH and DH and related problems are known today; see for example, Bombieri-Iwaniec [B-I], Heath-Brown [HE], Iwaniec [I], and Jutila [J]. In Chapters IV and V accounts of some of these techniques and developments are given. A detailed and clear treatment of Vinogradov's method and its consequences to giving the sharpest bounds towards the remainder term in the prime number theorem ($\sum_{p \leq x} 1 \sim x/\log x$, as $x \to \infty$) are described. A chapter is devoted to Selberg's theorem (to the effect that a positive proportion of the zeros are on the line $\mathrm{Re}(s) = 1/2$ and variations thereof (see Conrey [C] for a proof based on a method of Levinson that 40 percent are on the line). In Chapter VII the authors describe and prove what they call the "universality theorem": It asserts that for $|s| < 1/4$, $\log\zeta(s + 3/4 + iT)$ can be made to approximate any given analytic function by choosing T appropriately. This is a curious complex analytic property of $\zeta(s)$. In the same chapter other questions about the simultaneous distribution of $\zeta(\sigma + it)$ and its derivatives are derived. It would have been quite natural to include here another theorem of Selberg [S], which asserts that $\log\zeta(\frac{1}{2} + it)/\sqrt{\pi\log\log t}$ has a standard Gaussian limit distribution in $\mathbb{C}$ as $t \to \infty$.

A very interesting development, which is not discussed either here or in other recent books on $\zeta(s)$, is the discovery by Montgomery [MO1] and the calculations by Odlyzko [O] which show that the theory of the distribution of the eigenvalues of large random Hermitian matrices (Dyson [D], Mehta [Me]) apply remarkably well to the zeros of $\zeta(s)$.

The authors have made an extra effort to discuss the applications of some of the technical results on the zeta function to the distribution of prime numbers. Also more general L functions are discussed at various points in the book. This work is a valuable addition to the monographs on the zeta function. Other monographs on $\zeta(s)$ include the book by Edwards [E], which gives a nice historical account, and the book by Ivic [IV], which covers in depth the Density Hypothesis and Density Theorems as well as Mean Value Theorems. For accounts of the important developments in the theory of Dirichlet L-functions, see the monographs by Bombieri [B], Davenport [DA], Huxley [H], and Montgomery [MO2]. If I were limited to having just one book on my shelves on the Riemann zeta function, I would opt for the 1986 edition of Titchmarsh's monograph [T2].

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