BOOK REVIEW

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Free Lie algebras, by Christopher Reutenauer. Clarendon Press, Oxford, 1993, xvii+269 pp., $86.00. ISBN 0-19-853679-8

A *free Lie algebra* is a Lie algebra freely generated by a set called *alphabet*. The theory of free Lie algebras grew from the need of computing formally with commutators $[a, b] = ab - ba$ (using only the Jacobi identity and antisymmetry). It can be traced back to the turn of the century in the work of Campbell (1898), Poincaré (1899), Baker (1904), and Hausdorff (1906).

Given a set X (alphabet) and a ring K of coefficients, we can look at the immediate "structural relatives" of a free Lie algebra which are

(i) The free Lie algebra $L(X)$ (or $L_K(X)$)
(ii) The free magma $M(X)$
(iii) The free group $F(X)$
(iv) The algebra of noncommutative polynomials $K\langle X\rangle$
(v) The algebra of noncommutative formal power series $K\langle\langle X\rangle\rangle$

There are multiple connections among these concepts. For example, (iv) is the enveloping algebra of (i), Magnus's transformation is an embedding of (iii) in (v), and (i) is the graded Lie algebra associated with the lower central series of (iii).

The book reviewed here consists of nine chapters (plus an introductory Chapter 0) presenting the theory and its connections with

- Coding theory (for variable length codes)
- Control theory
- Hopf algebras
- Lie groups (exponentials, Hausdorff and Zassenhaus formulae)
- Iterated integrals
- Rewriting theory
- Enumerative and bijective combinatorics
- Symmetric group algebra and character theory
- Symmetric and quasisymmetric functions

The first part (Chapter 0) provides an exposition of the *minimum* prerequisites from the classical (and general) theory: Poincaré-Birkhoff-Witt theorem, Lazard elimination process, a first construction of linear bases for the free Lie algebra, realization as the subalgebra of noncommutative polynomials generated by the alphabet. Lazard sets are defined (these are sets that give rise to bases by successive application of Lazard's elimination process). These sets will be shown (in Chapter 4) to be exactly the Hall sets (defined in a completely different manner).

In Chapter 1, the Hopf algebra $K\langle X\rangle = \mathcal{U}(L(X))$ is presented. Four criteria are described for a noncommutative polynomial to be a Lie polynomial. Among them, one has, in particular, primitivity (known as Friedrich's criterion) and invariance by Dynkin's projection. A second law, the *shuffle product*, is defined. This law, denoted $\sqcup\!\sqcup$, is dual to the canonical coproduct of $\mathcal{U}(L(X))$. Combinatorially, it is the sum of the words w in which u and v are complementary subwords (this combinatorial characterization amounts to the etymology for the word *shuffle*).

Denoting by *conc* the *concatenation* of words and by c the comultiplication of $\mathcal{U}(L(X))$, we see that $(K\langle X\rangle\,,\ \mathrm{conc}\,,\ c\,,\ 1\,,\ \varepsilon)$ is now a bialgebra (in fact, like every enveloping algebra, it is a Hopf algebra, but the full structure is not needed here) and $\mathrm{End}(K\langle X\rangle)$ is an algebra with the *convolution law* given by

$$f * g(x) = \mathrm{conc}(f \otimes g)(c(x))\,.$$

Chapter 2 is devoted to the study of subalgebras (with Shirshov's theorem stating that each subalgebra of a free Lie algebra is itself a free Lie algebra—an analog of a theorem of Schreier for the free group) and automorphisms (in particular, a Jacobian condition for an endomorphism of the free Lie algebra to be an automorphism—similar to the theorem of Kurosh for the free nonassociative algebra).

Chapter 3, entitled "Logarithms and Exponentials," deals with the correspondence between $L(A)$ and the Hausdorff group $e^{L(A)}$ (indeed, the series of the form e^h, $h \in L(A)$, forms a group and can be characterized via the coproduct as a "grouplike" series).

The iterated integrals

$$\int_a^b dw$$

are defined. They will occur again in Chapter 6. The fact that the Chen series

$$\sum_{w\in A^*}\left(\int_a^b dw\right) w$$

is a grouplike element is proved here.

A direct sum decomposition of $K\langle A\rangle$ which will appear several times in the book is defined here by means of an n-ary operation, the symmetrized product

$$(P_1P_2\cdots P_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} P_{\sigma(1)}P_{\sigma(2)}\cdots P_{\sigma(n)}\,.$$

This permits one to give canonical complements of the increasing filtration of the enveloping algebra of $K\langle X\rangle = \mathcal{U}(L(X))$. More precisely, let

$$\mathcal{U}^n = 1 + L(X) + L(X)L(X) + \underbrace{L(X)L(X)\cdots L(X)}_{n \text{ times}}$$

(that is, polynomials which are the products of at most n Lie polynomials), and define U_n to be the linear span of the products $(P_1P_2\cdots P_n)$. Then

$$\mathcal{U}^{n-1} \oplus U_n = \mathcal{U}^n\,.$$

It follows that $K\langle X\rangle = \oplus_{n\geq 0} U_n$. Projections

$$\pi_n : K\langle X\rangle \to U_n$$

corresponding to this special decomposition are called canonical. It turns out that each projection will be identified with special elements of $K\mathfrak{S}_n$. In particular, π_1 allows one to express the coefficients of the Hausdorff series

$$H(a_1 a_2 \cdots a_p) = \log(e^{a_1} e^{a_2} \cdots e^{a_p}).$$

Chapters 4 and 5 are devoted to the construction and study of special (linear) bases of the free Lie algebra, namely, *Hall bases*.

The foliages (words obtained from the trees by the natural epimorphism $M(X) \to X^*$) of the trees of a Hall set (which are a generalization of classical Hall sets) form a totally ordered set of words (referred to, therefore, as *Hall words*). This set has the complete factorization property; i.e., each word w has a unique decreasing factorization:

$$w = h_1 h_2 \cdots h_n, \quad h_1 \geq h_2 \geq \cdots \geq h_n.$$

The author's approach is based on the construction of a rewriting system applied directly to the sequences of Hall trees. This has many advantages: the image can be considered in the monoid as well as in the free Lie algebra; this proves directly the Poincaré-Birkhoff-Witt factorization; it provides simple bases for the factors of the derived series of $L(X)$ (which is given in Chapter 5); and, moreover, although the results are deep, the exposition can be (and is) made elementary.

Here appears the first and strongest connection with the theory of (variable-length) codes. We will say a few words about these codes, since the reader may not be familiar with them.

Unlike the case of a free group, in a free monoid not all subsets generate a free substructure. A set which is a basis of a free submonoid (for example, $z(A - z)^*$, the words beginning with their unique occurrence of a given letter z), is called a *code*. Hall sets can be used to construct "variable length" codes with interesting synchronization properties (synchronizing word, comma-free property). The connection between Hall sets and Lazard's elimination process is described in the appendix.

The shuffle product endows $K\langle A\rangle$ with the structure of a commutative associative algebra. In fact, $(K\langle A\rangle, ⧢)$ is an algebra of polynomials with the Lyndon words as a transcendence basis. This, together with the study of subword functions and the combinatorics of subwords, is the subject of Chapter 6.

We have

$$\mu(w) = \sum_{v \in A^*} \binom{w}{v} v$$

where $\mu : F(A) \to K\langle\langle A\rangle\rangle$ is Magnus's transformation defined on the letters by $\mu(a) = 1 + a$. The combinatorial coefficients $\binom{w}{v}$ generalize the binomial ones, since $\binom{a^n}{a^p}$ is just the classical $\binom{n}{p}$. For fixed $u \in X^*$, the subword function

$$\varphi : g \to \binom{g}{u}$$

is all representatives (a function $\varphi : G \to K$ is called *representative* iff its shifts—i.e., the functions $\varphi_h : x \to \varphi(hx)$—span a finite-dimensional space).

The end of the chapter is devoted to showing properties of the asymptotic expansion of an element $g \in F(A)$

$$g \equiv \prod_{h \in H_{\leq N}} h^{n_h(g)} \mod(F_{N+1})$$

as an infinite product of powers of the Hall words (of limited order) with respect to the filtration of the lower central series. A theorem of Witt stating that the graded Lie algebra of the lower central series of $F(X)$ is $L_{\mathbb{Z}}(X)$ is given in the appendix. Also included in the appendix is additional information on the shuffle algebra, its connections with control theory, and recognizable subsets of the free group.

Hall words provide a section of conjugacy classes of words. These classes are called "circular words" (or "necklaces", since they can be drawn as necklaces with the letters as the pearls). "Circular Words" is the title of Chapter 7, in which primitive (i.e., without proper period) necklaces are studied with respect to enumeration and their relation with Hall words. Algorithms on Lyndon words and factorizations are given, as well as explicit bijections between multisets of primitive necklaces, words and irreducible polynomials over a finite field (the existence of such bijections is suggested by Witt's formulae).

The picture given at the beginning of this review (structural relatives and correspondences) survives under quotient by commutations of letters. This is the theory of *partially commutative structures* which is mentioned in the appendix, along with a formula for the determinant of a sum of matrices, Schützenberger's theorem on factorizations of a free monoid, and a result of the author and Berstel on cyclic languages.

Now comes, in the last two chapters, the representation theoretic aspect of the theory. Since a commutator defines a multilinear operation, there is a natural representation of $\mathrm{GL}(V)$ $(V = KA)$ on $L(A)$. Chapter 8 is devoted to the action of the symmetric group $\mathfrak{S}_A \subset \mathrm{GL}(V)$ on $L(A)$: its *characteristic* is computed (that is, in the Schur-Weil duality, the symmetric function which is the character of the corresponding representation). The splitting into irreducibles is given via the *major index*, a statistic on Young tableaux, which arises naturally in the representation on *covariants* (i.e., the ring of all polynomials factored by the ideal generated by the constantless symmetric functions). The computation of the character of this representation is given.

The symmetric group $\mathfrak{S}_n$ acts also on words of length n by *place permutation*. Explicitly,

$$(a_1 a_2 \cdots a_n)\sigma = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} .$$

An idempotent $e \in K\mathfrak{S}_n$ then defines a projection

$$K_n\langle A\rangle \to K_n\langle A\rangle e ,$$

and Lie idempotents are those such that $K_n\langle A\rangle e = L_n(A)$. There are several known Lie idempotents bearing the names of Dynkin, Klyachko, and also π_1 as described earlier. (Recently, Gel′fand et al. obtained all known idempotents from a one-parameter family of Lie idempotents [1].) All these idempotents (except the orthogonal idempotent discussed in the appendix) belong to a solvable subalgebra of $K\mathfrak{S}_n$, which is the subject of the last chapter—Solomon's descent algebra.

A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ is an index $1 \le i < n$ such that $\sigma(i) \ge \sigma(i+1)$. The set of descents of σ is denoted $D(\sigma)$. For $S \subseteq \{1, 2, \ldots, n-1\}$, we define the descent class

$$D_S = \sum_{D(\sigma)=S} \sigma .$$

Solomon showed (in fact, for any finite Coxeter group) that

$$D_S D_T = \sum n_{ST}^R D_R$$

with $n_{ST}^R \in \mathbb{N}$, so that the linear span of the descent classes is a subalgebra of $\mathfrak{S}_R$. Let Σ_n denote this algebra. The dimension of Σ_n is also the number of *compositions of* n (i.e., *vectors with positive integer coordinates* whose sum is n), which is 2^{n-1}.

The sum $\Sigma = \oplus_{n \geq 0} \Sigma_n$ embeds as a subalgebra Γ of $\mathrm{End}(K\langle a \rangle)$ ($\mathfrak{S}_n$ acts by zero on $K_m\langle A \rangle$ if $m \neq n$) which is stable under the convolution product defined above. Reutenauer's exposition starts with convolution and makes very clear that composition in Σ and in Γ are opposites. These algebras are antiisomorphic, and the correspondence is expressed by means of bases indexed by compositions. It turns out that $(\Gamma_n, \circ)$ is solvable. In fact, a triangular decomposition of this algebra is given by means of projections

$$\pi_\lambda : K\langle A \rangle \to K\langle A \rangle$$

that refine the π_n. We have (all indices have weight n)

$$\Gamma_n = (\oplus_\lambda K.\pi_\lambda) \oplus (\oplus_{\lambda > \mu} \pi_\mu \Gamma_n \pi_\lambda)$$

where $\geq$ is an order on *partitions* (decreasing compositions) such that $\pi_\mu \pi_\lambda \neq 0 \Rightarrow \lambda \geq \mu$. Note that in [1] another realization of the algebra $\oplus \Sigma_n$ is given in terms of *noncommutative symmetric functions*.

In closing, one considers an algebra larger than the algebra of symmetric power series. This is the algebra of *quasisymmetric functions* over a totally ordered alphabet $(X, <)$. This algebra is generated by the *monomial quasisymmetric functions*

$$M_C = \sum_{x_1 < x_2 < \cdots < x_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

where $C = (i_1, i_2, \ldots, i_k)$ is a composition. It is true that $M_{C_1} M_{C_2} = \sum M_{C_i}$, so that the monomial functions form a linear basis of the algebra (in fact, this algebra can be shown to be the Grothendieck ring of a suitable category, the product corresponding to the tensor product). The ring of quasisymmetric functions is dual to the descent algebra by means of the formula

$$F_{C(\sigma)}(XY) = \sum_{\sigma = \beta\alpha} F_{C(\alpha)}(X) F_{C(\beta)}(Y),$$

where XY is the lexicographic product of the totally ordered sets $(X, <)$, $(Y, <)$, and $C(\sigma)$ is the integer vector of the "jumps" of the descent set of σ. These quasisymmetric functions are used to give generating series of special sets of permutations.

The book by Christopher Reutenauer is very well written: proofs are as elementary as they can be, and examples are given to illustrate the notions that are introduced progressively and at the right moment. We are indebted to him for having put some order in the welter of properties and connections surrounding the free Lie algebras. The book is recommended for everyone, whether familiar or less than familiar with the subject, for pleasure and for knowledge.

1. I. M. Gel′fand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J. Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. (to appear).

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