BOOK REVIEW

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Clifford algebra and spinor-valued functions, a function theory for the Dirac operator, by R. Delanghe, F. Sommen, and V. Souček. Kluwer, Dordrecht, 1992, xvi + 485 pp., $176.00. ISBN 0-7923-0229-X

The seminal paper in which W. K. Clifford introduced the algebras which now bear his name appeared in the first volume of the *American Journal of Mathematics* in 1878 [C]. The following year, at age thirty-three, Clifford died of tuberculosis. He was professor of applied mathematics at University College, London, and had previously been a student of James Maxwell. This paper is quite modern in its attempt to relate geometry and algebra, doing more than just presenting a higher-dimensional version of complex numbers and quaternions.

Part of the reason for the early development of Clifford algebra was the desire to understand the mathematical basis of Maxwell's equations. Much later, and independently, Dirac used a spinor representation of a particular Clifford algebra as a framework for his famous equations describing a quantum theory of electrons [D].

The 2^n–dimensional Clifford algebra $\mathbb{R}_{(n)}$ can be described as the vector space with basis $\{ e_S \}$ indexed by the subsets S of $\{ 1 , 2 , \ldots , n \}$ and with algebraic structure defined by

$$\begin{aligned} e_\varnothing &= 1 , && \text{the identity} \\ e_j &= e_{\{ j \}} && \text{if } 1 \le j \le n \\ e_j e_k + e_k e_j &= -2\, \delta_{j,k} && \text{if } 1 \le j \le k \le n \\ e_{j_1} e_{j_2} \ldots e_{j_s} &= e_{\{ j_1 , j_2 , \ldots , j_s \}} && \text{if } 1 \le j_1 < j_2 < \cdots < j_s \le n . \end{aligned}$$

The product of $u = \sum_S u_S e_S$ and $v = \sum_T v_T e_T$ is $uv = \sum_{S,T} u_S v_T e_S e_T \in \mathbb{R}_{(n)}$, where u_S and v_T are real numbers.

The Clifford algebra $\mathbb{R}_{(n)}$ contains $\mathbb{R} \oplus \mathbb{R}^n$ as a vector subspace, when $\lambda e_\varnothing + \sum_{j=1}^n \xi_j e_j \in \mathbb{R}_{(n)}$ is identified with $(\lambda , \xi_1 , \xi_2 , \ldots , \xi_n) \in \mathbb{R} \oplus \mathbb{R}^n$. It can be complexified in the usual way to obtain $\mathbb{C}_{(n)}$. Clifford algebras based on an indefinite metric $g_{j,k}$ in place of $\delta_{j,k}$ are also important.

The Clifford algebras $\mathbb{R}_{(0)}$, $\mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ can be identified with the real numbers, the complex numbers and the quaternions, respectively. In these algebras, every non–zero element has an inverse. Although this is not the case in higher dimensions, it is nevertheless true that every non–zero vector $\xi = \sum_{k=1}^n \xi_k e_k$ has an inverse in $\mathbb{R}_{(n)}$, namely, $\xi^{-1} = -\xi \, |\xi|^{-2}$. In other words, $\xi^2 = -|\xi|^2$. This is possibly the most important property of these Clifford algebras and the key fact used in developing Clifford analysis.

Indeed if $\mathbf{D}$ denotes the Dirac operator,

$$\mathbf{D} \;=\; \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\, e_j \;\;,$$

then its square is the negative of the Laplacian $\boldsymbol{\Delta}$, namely,

$$\mathbf{D}^2 \;=\; -\boldsymbol{\Delta} \;=\; -\sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2} \;\;.$$

These operators act on functions which are defined on open subsets Ω of $\mathbb{R}^n$ taking their values in a Clifford module $\mathcal{X}$, which could be $\mathbb{R}_{(n)}$ itself. Alternatively, $\mathcal{X}$ could be a representation of $\mathbb{R}_{(n)}$ such as the Dirac spinors.

A smooth function u which satisfies $\mathbf{D}u = 0$ is called a monogenic function (or sometimes a Clifford holomorphic or a Clifford regular function) and is clearly harmonic. The theory of monogenic functions is called Clifford analysis. It is amazing how much of complex analysis generalises to this setting. Basic results were independently discovered and rediscovered several times this century, though now the subject has a life of its own. Some of the earlier papers are by Dixon [Dix], Moisil and Theodorescu [MT], Fueter [F] and Iftimie [I].

The first book on Clifford analysis appeared in 1982, written by Brackx, Delanghe and Sommen [BDS]. In this book and in related papers, the authors developed the basic theory of this subject. The book remains the basic reference work on this topic. Other books on related material are those by Gilbert and Murray [GM], Gürlebeck and Sprössig [GS], Lawson and Michelsohn [LM], Boos-Bavnbeck and Wojciechowski [BBW] and Mitrea [M].

The principal result from complex analysis which generalises to Clifford analysis is the Cauchy integral formula. The higher-dimensional version states that every monogenic function u can be represented as

$$u(x) \;=\; \frac{1}{\sigma_{n-1}} \int_{b\Omega} \frac{x-y}{|x-y|^n}\; \nu(y)\, u(y)\, dS_y \;\;,$$

for $x \in \Omega$, where $b\Omega$ is the smooth boundary of a bounded open subset Ω of $\mathbb{R}^n$ and $\nu(y)$ is the exterior unit normal at $y \in b\Omega$. The first two terms in the integrand are multiplied according to the definition of the Clifford algebra and then act on the element $u(y)$ of the Clifford module $\mathcal{X}$. Here σ_{n-1} is the $(n-1)$ dimensional volume of the unit sphere in $\mathbb{R}^n$.

Higher-dimensional analogues of many results from complex analysis can be obtained by using this formula. For example, the generalisation of Liouville's theorem states that the only monogenic functions which are defined and bounded on all of $\mathbb{R}^n$ are the constant functions. Also of interest is the fact that every real analytic function defined on an open subset of $\mathbb{R}^{n-1}$ has a monogenic extension to a neighbourhood in $\mathbb{R}^n$.

In recent years considerable interest has been shown in developing various aspects of Clifford analysis, both for its own sake and for its applications. For example, Ryan has discovered the equivalent of "domains of holomorphy" in this context [R2] and has proved an analogue of Runge's approximation theorem [R1] for domains in $\mathbb{C}^n$. There are significant recent developments of Clifford analysis related to

Cauchy integrals, Plemelj formulae and harmonic analysis on Lipschitz surfaces, and applications to harmonic functions on Lipschitz domains. There are also Clifford wavelets, Clifford martingales, the Clifford $T(b)$ theorem, and other connections with harmonic analysis. For material of this nature, see the books [GM], [GS], [M] mentioned above and papers such as [Mu], [DJS], [M^c], [AT], [LM^cS], [GLQ] and [LM^cQ]. There are survey lectures of mine [M^cI] on some such matters in the *Proceedings of the Conference on Clifford Algebras in Analysis*, held in Fayetteville, Arkansas, in 1993, as well as papers by other participants.

Let us now turn to the book under review. The series editor states that, "Here then is the complete up-to-date account of the subject [Clifford analysis] by the experts and originators themselves." The authors, however, do not regard completeness as their brief, but rather concentrate on various areas in which they and their collaborators have played a role since the appearance of the book [BDS].

For the most part, they treat this material well, the highlights being Chapters IV and V on monogenic differential forms and the Penrose transform.

The first chapter (and the zero'th chapter) contains the algebra which is needed in the remainder of the book. There is a great deal of basic information about Clifford algebras, the Pin and Spin groups, the representations of Clifford algebras in the spaces of Dirac and Weyl spinors, and the Fock space. The reader may be interested in looking also at the 1957 lectures of M. Riesz which have recently been republished [R].

In the second chapter, monogenic functions are introduced, and relevant results from the book [BDS] are summarised. This is followed by an extensive treatment of spherical monogenic functions and related topics. The decomposition of a function defined on the unit sphere into spherical monogenics, rather than into spherical harmonics, is more descriptive and will, I am sure, have many applications. Such a decomposition was given in the quaternionic case by Sudbery [S]. The third chapter is on special functions. It has sections on Gegenbauer and Hermite polynomials, the Cauchy–Kovalevskaya method, Cauchy type integrals, plane wave integrals, and Riesz potentials.

The material in the first three chapters is presented in a clear, extremely detailed fashion, which is easy to read page by page, though sometimes difficult to skim through and hunt for information, as definitions and results are often not highlighted. A more detailed index and a notation list would have helped considerably.

In Chapters IV and V, the presentation becomes tighter, and the mathematics deeper. Some familiarity with homology, co-homology and related matters is expected here, though not too much, as the presentation is extremely good, and clear statements are given of the background material which is needed. There are appendices on Leray–Norguet residue theory and homogeneous vector bundles.

Chapter IV is devoted to monogenic differential forms and residues. I found this material fascinating. The authors succeed in generalising the theory of holomorphic forms to higher dimensions in such a way that (i) there is invariance with respect to the group $\mathrm{Spin}(m)$, (ii) Cauchy's theorem holds, and (iii) the homology of the complex describes faithfully the topology of the domain $\Omega \subset \mathbb{R}^n$. This takes some care, because if the spaces in the complex are too small, then there is no Cauchy theorem, while if they are too large, then the homology is lost.

In Chapter V, the authors develop the connection between Clifford analysis and the Penrose transform, with particular reference to the generalisation of the Penrose transform developed recently by Baston and Eastwood [BE]. This appears to be significant research, treated here for the first time.

The book ends with an appendix on computer algebra programs for Clifford algebras written in Reduce. There is software for these programs on a computer disk which comes with the book.

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