BOOK REVIEW

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Barrelledness, Baire-like and (LF)-spaces, by Michael Kunzinger. Pitman Research Notes in Mathematics Series, vol. 298, Longman Scientific & Technical, Harlow, Essex, copublished with Wiley, New York, 1993, xiii+160 pp., $46.95. ISBN 0-582-23745-9

This volume of research notes is concerned with a theory that has evolved over the last twenty years or so. It presents a fairly complete account of the results that have been obtained on classes of locally convex spaces (l.c.s.) which include all Baire spaces and are contained in the class of all barreled (or tonnelé) spaces. (Nobody seems to follow the reviewer's Webster spelling of the word "barreled", but so be it.) Every beginning student of abstract analysis is exposed to the concepts of Banach spaces and Frechet spaces (or (F)-spaces, i.e., complete metrizable l.c.s.) and, in addition to Hahn-Banach, the Principle of Uniform Boundedness (P.U.B), and the Open Mapping and Closed Graph Theorems. Those are shown to me valid because the underlying spaces are Baire (and local convexity dispensable). A Baire space is a topological space in which every non-void open set is of second category, but what is a barreled l.c.s.? There are many (equivalent) definitions, but to me the most revealing seems to be that barreled l.c.s's are those for which the P.U.B. ("Every pointwise bounded set of continuous linear maps into any l.c.s. is equicontinuous") holds by fiat. The success of this definition, of course, lies in how far this class transcends Baire spaces and what its permanence properties are under the standard constructions of projective and inductive l.c. topologies. A good parallel in topology is given by the notion of compactness: Compact (i.e., closed and bounded) subsets of the real line were known, some one hundred years ago, to have the Heine-Borel covering property; making this the definition of compactness was immensely fruitful, as is well known.

These remarks are intended to illuminate the motivation for the research covered by this book: finding and studying classes of l.c. spaces "bracketed" by Baire spaces at the special end and by barreled spaces at the general end, with a view to permanence and the validity of open mapping/closed graph theorems; the results turn out to be very useful in the attempt to classify (LF)-spaces (see below). To this reviewer, a strong motivation for the content of the book is desirable, since at first glance the host of required definitions and ensuing ramifications is not very appealing.

Before trying to survey the seven chapters of the book, we must mention several earlier works on which, in addition to research papers, the presentation is based. Those are especially the monographs by Bonet-Perez Carreras [BP] on barreled spaces, by Valdivia [V] on topics in locally convex spaces, and the major article by Diestel-Morris-Saxon [DMS] on varieties of linear topological spaces. The proofs given in this volume are complete but draw heavily on the reviewer's book [S] as well as that by Horvath [H]. Armed with these, an advanced student should be able to master the text.

We now proceed to survey the seven chapters in Kunzinger's work.

Chapter 1 presents some common (e.g., infrabarreled, evaluable) and some not-so-common (e.g., infra-evaluable, ω-barreled, ω-evaluable, c_0-barreled) weakenings of the notion of barreledness. Various tools are developed for later use, and some interesting results on subspaces of countable codimension are given. (Most classes of spaces considered later are stable under the operation of taking such subspaces.) For example, if E is a Mackey space with weak*-sequentially complete dual and M is a closed subspace of countable codimension, then every algebraic complement of M is topological and carries the strongest l.c. topology (Saxon). Examples and references conclude this and all subsequent chapters.

Chapter 2 discusses the strongest l.c. topology (on a real or complex vector space). The direct l.c. sum Φ_m of m (any cardinal) copies of the scalar field K is studied from various angles, e.g., from that of being contained in a given l.c.s. E. The importance of Φ_m (especially for $m = \aleph_0$, then denoted by Φ) here stems from its usefulness of characterizing almost-Baire and (LF)-spaces later on.

Chapter 3 provides some fundamental theorems of the theory of varieties of l.c.s. due to [DMS]. The author intends the concept of variety to be a unifying element in his work—namely, varieties being classes of l.c.s. closed under the operations of taking subspaces, separated quotients, Cartesian products, and isomorphic images. Many of the results presented seem not as widely known as they deserve to be. Example: The variety of all Schwartz spaces and the variety of all nuclear spaces both have universal generators.

Chapter 4 now approaches the main topic through a thorough discussion of (linear topological) Baire spaces. Thus a t.v.s. is Baire if and only if every absorbing, balanced, and closed subset is not rare. Also included are two major results by Arias de Reyna (1980, 1982): There exist normed Baire spaces whose product is not Baire, and—assuming Martin's Axiom—a negative answer to the so-called Wilansky-Klee conjecture (i.e.: on a Banach space, a linear form is continuous if and only if its kernel is of first category).

Chapter 5, on unordered Baire-like and (db)-spaces, considers several weakenings of the Baire property (within the realm of l.c.s.) that have produced results of independent interest (Valdivia, Robertson, Saxon, Todd et al.) but also lead to a deeper understanding of Baire spaces and their failure with respect to permanence. The basic idea is: Generalize the Baire property by requiring the l.c.s. E not to be coverable by the union of a sequence (increasing if marked (i)) of subsets having property (P). The space is then called

(a) convex-Baire if $P =$ rare, convex;
(b) unordered Baire-like if $P =$ rare, absolutely convex;
(c) (db)-space if (i) and $P =$ non-dense or non-barreled linear subspace;
(d) Baire-like if (i) and $P =$ rare, absolutely convex;
(e) quasi-Baire: E is barreled and (i) with $P =$ rare subspace.

These classes are all distinct, and one of the major results seems to be that the properties of being Baire or barreled, plus (b), (d), (e) above are inherited from any product by the subspace of all countably non-zero vectors. Inclusions are: Baire $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ barreled.

Chapter 6 discusses Baire-like and quasi-Baire spaces. Major results are a very general open mapping—closed graph theorem (6.1.8), the stability of both classes under arbitrary products and their interrelation with Φ. Example: A barreled space is quasi-Baire if and only if it does not contain Φ complemented.

Finally, Chapter 7 applies much of the preceding material to (LF)-spaces, here defined to be inductive limits of a strictly increasing sequence of (F)-spaces. The greater part of the chapter is concerned with a classification, due to Narayanaswami and Saxon, of (LF)-spaces into three classes according to whether the space E is metrizable, E is not metrizable and has a defining sequence with some member dense in E, or E has a defining sequence with no member dense in E. These classes can be characterized in a manner reminiscent of the geometry of Banach spaces, if not of its results. For example, a space E belongs to the first-mentioned class if and only if it does not contain Φ. The classes also serve to provide general examples to show that the classes introduced in Chapter 5 are all distinct. Other applications concern the "separated quotient problem", still open for Banach spaces.

What, now, is the overall impression the book leaves? Certainly, many of the results collected here do not lend themselves to stunning or casual applications, and the reader only marginally familiar with topological (in particular, locally convex) vector spaces will probably gain very little, as is the case with most highly specialized texts. But the manuscript is very carefully written, rather well organized, and for those interested in or working with Baire-like spaces (in the non-technical sense), it will be very valuable.

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