# BOOK REVIEW

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*Invariant potential theory in the unit ball of* $\mathbb{C}^n$, by Manfred Stoll. London Math. Soc. Lecture Note Ser., vol. 199, Cambridge University Press, London and New York, 1994, x + 173 pp., $29.95. ISBN 0-521-46830-2

To one who first met potential theory almost half a century ago it comes as a surprise that central topics such as "equilibrium distribution" and "energy integral" are never mentioned in this book. Even "capacity" seems to occur in only one short section toward the end. The content of mathematical subjects obviously changes with time. "Harmonic analysis" is another example: this term used to refer to trigonometric series (via overtones), was then expanded to cover the study of functions on groups or of function spaces that were invariant under the action of some group, and nowadays mathematicians who call themselves harmonic analysts seem to spend most of their time investigating all sorts of maximal functions.

The book under review is primarily concerned with the *invariant Laplacian* $\tilde{\Delta}$ on the unit ball $B$ of $\mathbb{C}^n$ and with the *invariantly harmonic* (also called $\mathcal{M}$-*harmonic*) functions $f$ which satisfy $\tilde{\Delta} f = 0$. These will be defined presently. (To be quite explicit, $B$ is the set of all $z = (z_1, \cdots, z_n)$ in $\mathbb{C}^n$ with $\sum_1^n |z_i|^2 < 1$.)

There are at least three local notions of harmonicity that occur naturally in the study of several complex variables.

(i) $u$ is *harmonic* in an open set $\Omega \subset \mathbb{C}^n$ if $\Delta u = 0$, where

$$\Delta = \sum_1^n \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) = 4 \sum_1^n \frac{\partial^2}{\partial z_i \partial \overline{z}_i}$$

is the ordinary Laplacian.

(ii) $u$ is *n-harmonic* if $\partial^2 u / \partial z_i \partial \overline{z}_i = 0$ for $i = 1, \ldots, n$, i.e., if $u$ is harmonic in each variable separately.

(iii) $u$ is *pluriharmonic* if $\partial^2 u / \partial z_j \partial \overline{z}_k = 0$ for all $j, k = 1, \ldots, n$. Locally, the real-valued pluriharmonic functions are precisely the real parts of holomorphic functions.

In certain regions $\Omega \subset \mathbb{C}^n$ it also makes good sense to introduce some global definitions.

Let $\mathrm{Aut}(\Omega)$ be the group (relative to composition) of all one-to-one holomorphic maps of $\Omega$ onto $\Omega$. If $\mathrm{Aut}(\Omega)$ is transitive on $\Omega$, then $\Omega$ is called *homogeneous*. If, in addition, there is a $\psi \in \mathrm{Aut}(\Omega)$ that has period 2 $(\psi(\psi(z)) = z)$ and fixes exactly one point of $\Omega$, then $\Omega$ is called *symmetric*.

Choose $a \in B$, let $P$ be the orthogonal projection of $\mathbb{C}^n$ onto the subspace [a] generated by $a$, let $Q = I - P$ be the projection onto the orthogonal complement of [a] in $\mathbb{C}^n$, and define

$$\varphi_a(z) = \frac{a - P_z - \sqrt{1 - |a|^2} Qz}{1 - \langle z , a \rangle} \qquad (z \in B) \tag{1}$$

where $\langle z , a \rangle = \sum z_i \overline{a}_i$. It is easy to prove ([6, Theorem 2.2.2]) that $\varphi_a \in \mathrm{Aut}(B)$, that $\varphi_a$ interchanges $0$ and $a$, and that $\mathrm{Aut}(B)$ is therefore transitive on $B$. The map $z \to -z$ shows then that $B$ is a *bounded symmetric domain*.

The polydiscs are other well-known examples of such domains. Elie Cartan [1] classified them all.

To define $\tilde{\Delta} u$, for $u \in C^2(B)$, let $(\tilde{\Delta} u)(a)$ be the ordinary Laplacian of $u \circ \varphi_a$, evaluated at the origin, i.e.,

$$(\tilde{\Delta} u)(a) = (\Delta(u \circ \varphi_a)(0). \tag{2}$$

It is then almost immediate ([6, Theorem 4.1.2]) that $\tilde{\Delta}$ is *invariant* in the sense that it commutes with the action of $\mathrm{Aut}(B)$:

$$\tilde{\Delta}(u \circ \psi) = (\tilde{\Delta} u) \circ \psi \tag{3}$$

for every $\psi \in \mathrm{Aut}(B)$.

Thus $\tilde{\Delta} u = 0$ implies $\tilde{\Delta}(u \circ \psi) = 0$ for every $\psi \in \mathrm{Aut}(B)$. Since (1) is reminiscent of a Möbius transformation, these $u$'s are called $\mathcal{M}$*-harmonic*.

The author does not define $\tilde{\Delta}$ as in (2), but takes the reader on a longer tour which places the matter into a wider context:

To every bounded domain $\Omega \subset \mathbb{C}^n$ corresponds its Bergman kernel $K(z , w)$, whose characteristic features are as follows. Define

$$K[f](z) = \int_\Omega f(w) K(z , w) dV(w) \qquad (z \in \Omega), \tag{4}$$

for $f \in L^1(\Omega)$ with respect to Lebesgue measure $dV$. Then $K[f] = f$ (reproducing property) for every holomorphic $f \in L^1(\Omega)$, and $f \to K[f]$ is the orthogonal projection of $L^2(\Omega)$ onto its subspace of holomorphic functions. Moreover, $K(z , z) > 0$, and the matrix $(g_{ij}(z))(i , j = 1 , \cdots , n)$ defined by

$$g_{ij}(z) = \frac{\partial^2 \log K(z , z)}{\partial z_i \partial \overline{z}_j} \tag{5}$$

turns out to be positive definite. Denote its inverse by $(g^{ij}(z))$, and let $g = \det(g_{ij})$.

The $g_{ij}$'s determine an invariant metric on $\Omega$ which, in turn, induces the so-called *Laplace-Beltrami operator*

$$\tilde{\Delta}_\Omega = \frac{2}{g} \sum_{i , j=1}^{n} \left\{ \frac{\partial}{\partial \overline{z}_i} \left( g g^{ij} \frac{\partial}{\partial z_j} \right) + \frac{\partial}{\partial z_j} \left( g g^{ij} \frac{\partial}{\partial \overline{z}_i} \right) \right\} \tag{6}$$

which has the invariance property

$$\tilde{\Delta}_\Omega (f \circ \psi) = (\tilde{\Delta}_\Omega f) \circ \psi \tag{7}$$

for every $\psi \in \mathrm{Aut}(\Omega)$. (The author refers to [4] for the proof of (7).)

It should be pointed out here that (6) makes sense in every $\Omega$ but that (7) may well be true vacuously, because most regions $\Omega$ have no nontrivial automorphisms.

In $B$, the Bergman kernel is

$$K(z\,,\,w)=\frac{n!}{\pi^n(1-\langle z\,,\,w\rangle)^{n+1}} \tag{8}$$

and (6) becomes

$$\tilde{\Delta}_B=\frac{4(1-|z|^2)}{n+1}\sum_{j\,,\,k=1}^{n}(\delta_{jk}-z_j\overline{z}_k)\frac{\partial^2}{\partial z_j\partial\overline{z}_k}. \tag{9}$$

Starting from (2), it is quite easy ([6, Theorem 4.1.3]) to arrive at

$$\tilde{\Delta}=(n+1)\tilde{\Delta}_B. \tag{10}$$

These two operators thus annihilate the same functions.

Returning to an arbitrary $\Omega$, a function $u\in C^2(\Omega)$ is said to be *weakly harmonic* if $\tilde{\Delta}_\Omega u=0$, and to be *strongly harmonic* if $Du=0$ for every linear differential operator $D$ which satisfies $D1=0$ and $(Df)\circ\psi=D(f\circ\psi)$ for all $\psi\in\mathrm{Aut}(\Omega)$.

The author states, without proof or reference (pp. 3, 31)

(a) that these two concepts coincide when $\Omega=B$, and
(b) that this happens *only* when $\Omega=B$.

The proof of (a) is actually quite easy. If $\tilde{\Delta}f=0$, then $f$ has the mean-value property

$$f(z)=\int_{\mathcal{U}}f(\varphi_z Uw)dU \qquad ((z\,,\,w)\in B\times B) \tag{11}$$

where $dU$ denotes the Haar measure on the unitary group $\mathcal{U}$; the left side of (11) is $(f\circ\varphi_z)(0)$, the right side is the average of $f\circ\varphi_z$ over the sphere of radius $|w|$, center $0$. Moreover, (11) forces $f$ to be in $C^\infty(B)$. If now $D$ is as above, fix $z$ in (11), apply $D$ to the resulting function of $w$, and evaluate the result at $w=0$, getting

$$0=\int_{\mathcal{U}}(Df)(\varphi_z Uw)dU\Big|_{w=0}=(Df)(\varphi_z(0))=(Df)(z). \tag{12}$$

As for (b), I don't know how to prove it.

On *polydiscs*, the strongly harmonic functions are precisely the $n$-harmonic ones. There is an example on page 24 of a weakly harmonic function which is not strongly harmonic.

When $n=1$, i.e., when $B$ is the unit disc in $\mathbb{C}$, then the harmonic functions are the same as the $\mathcal{M}$-harmonic ones, even though $\tilde{\Delta}\neq\Delta$; this follows from (9). But when $n>1$, there are significant differences. Here are three:

(i) It is often very convenient to replace a harmonic function $u$ by its dilates $u_r$ $(u_r(z)=u(rz))$ and then let $r\nearrow 1$. This device is not available for $\mathcal{M}$-harmonic functions. In fact, if there is one $r$, $0<r<1$, such that $u$ and $u_r$ are $\mathcal{M}$-harmonic, then $u$ is pluriharmonic ([6, Theorem 4.4.10]). This makes it much harder to prove that even *bounded* $\mathcal{M}$-harmonic functions are Poisson integrals of their boundary values. (See Theorem 5.8 of Stoll's book, or [3], or David Ullrich's proof of Theorem 4.3.3 in [6].)

(ii) If $\Delta u=0$ and $\tilde{\Delta}u=0$, then $u$ is pluriharmonic.

(iii) In PDE language, $\Delta$ is *uniformly elliptic*, whereas $\tilde{\Delta}$ degenerates at the boundary $S$ of $B$. This accounts for the following phenomenon (pp. 48–51): There are $C^\infty$-functions on $S$ whose $\mathcal{M}$-harmonic extension to $\overline{B}$ is not $C^\infty$. In fact, smoothness at the boundary forces $\mathcal{M}$-harmonic functions to be pluriharmonic! The basic idea of this is in [2]. A sharp (unpublished) result in this direction involves the radial derivative

$$(\mathcal{D}u)(r\zeta) = \lim_{t\to r} \frac{u(t\zeta) - u(r\zeta)}{t - r} \qquad (0 < r < 1\,,\, \zeta \in S) \tag{13}$$

and says:

*If* $(\tilde{\Delta}u)(z) = 0$ *for all* $z$ *in the unit ball of* $\mathbb{C}^n$, $\epsilon(r) \to 0$ *as* $r \to 1$, *and*

$$\int_S |(\mathcal{D}^n u)(r\zeta)|^2 d\sigma(\zeta) = \epsilon(r) \log^2\left(\frac{1}{1-r}\right) \qquad (0 < r < 1)\,, \tag{14}$$

*then* $u$ *is pluriharmonic.*

Here $\sigma$ is the rotation-invariant positive measure on $S$ that gives $\sigma(S) = 1$.

A good part of the book is devoted to $\mathcal{M}$-*subharmonic* functions. These are upper semicontinuous and, by definition, are characterized by the invariant mean-value inequality

$$f(z) \le \int_S f(\varphi_z(r\zeta)) d\sigma(\zeta). \tag{15}$$

To every such $f$ corresponds a unique regular Borel measure $\mu_f$ (the *Riesz measure* of $f$) which satisfies

$$\int_B h d\mu_f = \int_B f \tilde{\Delta} h d\lambda \tag{16}$$

for every smooth $h$ with compact support in $B$. Here

$$d\lambda(z) = n!\pi^{-n}(1 - |z|^2)^{-n-1} dV(z) \tag{17}$$

which has the invariance property $\lambda(\varphi(E)) = \lambda(E)$ for all $\varphi \in \mathrm{Aut}(B)$.

The fundamental solution (Green's function) for $\tilde{\Delta}$ is

$$G(z\,,\, w) = g(\varphi_w(z)) \qquad ((z\,,\, w) \in B \times B) \tag{18}$$

where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1 - t^2)^{n-1} t^{1-2n} dt. \tag{19}$$

The *Green potential* of a nonnegative measure $\mu$ on $B$ is

$$G_\mu(z) = \int_B G(z\,,\, w) d\mu(w) \qquad (z \in B). \tag{20}$$

The finiteness of $\int_B (1 - |w|^2)^n d\mu(w)$ is necessary and sufficient for $G_\mu(z) \not\equiv +\infty$. In that case one has an extension of Littlewood's theorem [5], namely, (p. 96)

$$\lim_{r\nearrow 1} G_\mu(r\zeta) = 0 \qquad \text{a.e. on } S. \tag{21}$$

The classical Riesz decomposition theorem takes the following form (p. 70):

*If $f$ is $\mathcal{M}$-subharmonic and has an $\mathcal{M}$-harmonic majorant in $B$, then*

$$f(z) = H_f(z) - \int_B G(z, w) d\mu_f(w) \tag{22}$$

*where $H_f$ is the least $\mathcal{M}$-harmonic majorant of $f$ in $B$, and $\mu_f$ is as in* (16).

The original proofs of (21) and (22) are in [7].

The book contains also a number of more technical results concerning the boundary behavior of Green potentials, of Hardy spaces, of Bergman spaces, and of Dirichlet spaces. Many of these are due to the author.

The topics included in this book are well chosen and well presented. The index is a bit skimpy, and a one-page list of special notations would have been helpful. Among the few misprints only one seems to have any mathematical implication: at the bottom of page 35, if both functions are only *locally* in $L^1$ and in $L^p$, their convolution need not be defined.

And surely Shur should be Schur!

## References


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