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Solution of continuous nonlinear PDEs through order completion, by M.B. Oberguggenberger and E.E. Rosinger, North-Holland Math. Stud., vol. 181, North-Holland, Amsterdam, London, and New York, 1994, xvi + 432 pp., \$143.00, ISBN 0-444-82035-3

The authors propose a new definition of generalized solutions to arbitrary nonlinear partial differential equations. It is based on a construction of a new space of generalized functions.

Let Ω be a domain in $\mathbf{R}^n$. It is well-known that the space of distributions $\mathcal{D}'(\Omega)$ can be defined in the following way. Consider the set of sequences $\{u_k\}$ of $C_0^\infty(\Omega)$ functions such that for any $\varphi \in C_0^\infty(\Omega)$ there exists a finite $\lim_{k\to\infty} \int u_k(x)\varphi(x)dx$. Two sequences $\{u_k\}$ and $\{v_k\}$ are said to be equivalent if

$$\lim_{k\to\infty} \int (u_k(x) - v_k(x))\varphi(x)dx = 0.$$

The class of equivalent sequences is called a distribution.

The distribution theory is a very powerful tool for studying the linear partial differential equations with smooth coefficients. But it becomes useless when one has to regard a nonlinear problem. Besides, even very nice linear equations often have no solutions in the distribution space. For example, the following classical linear equations have no distribution solutions in any neighborhood Ω of the origin for a large set of functions f from $C_0^\infty(\Omega)$:

$$\frac{\partial u}{\partial x} + ix\frac{\partial u}{\partial y} = f(x,y),$$

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial y^2} = f(x,y),$$

$$\frac{\partial^2 u}{\partial x^2} - a^2(x)\frac{\partial^2 u}{\partial y^2} + b(x)\frac{\partial u}{\partial y} = f(x,y)$$

with some smooth functions a and b. Note that the first equation is elliptic for $x \neq 0$, the second is parabolic, and the third one is hyperbolic.

Dealing with a boundary value problem for a differential equation, one can use an approximation of data by smooth functions. Usually for well-posed linear problems the result does not depend on the choice of the approximation. However, this is not in general true for nonlinear problems. For example, the product $\delta \cdot \delta$ of two Dirac functions $\delta(x)$ is not defined in the theory of distributions (see [6]). Sometimes physicists define this product as $C\delta(x)$, where C is an arbitrary constant. Really, using two suitable δ-like sequences $\delta_k(x)$ and $\delta'_k(x)$ one can prove that $\delta_k(x)\delta'_k(x) \to C\delta(x)$ in the sense of the distribution theory. This means that there are many different δ-functions and the result can depend on the choice of the approximations.

Recently J.-F. Colombeau [1], [2] constructed the spaces $\mathcal{G}(\Omega)$ of generalized functions, which contain the distribution space $\mathcal{D}'(\Omega)$ and which are an algebra. His main idea is that a generalized function defined as a limit in some sense of a

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sequence of smooth ones can be identified with this sequence. A similar construction has been proposed also by E. E. Rosinger; see, for example, [5].

I will state here briefly my variant of this theory [3], which seems to be more general and simpler than that of Colombeau.

Consider the set of all sequences of functions u_k from $C^\infty(\Omega)$. Two such sequences u_k and u'_k are called equivalent if there is a number N such that $u_k = u'_k$ in Ω for $k > N$. The class of equivalent sequences is called a generalized function. It is easy to see that the space $\mathcal{G}(\Omega)$ of generalized functions contains the distribution space $\mathcal{D}'(\Omega)$ and is an algebra. Moreover, the composition of two generalized functions is a generalized function, and a derivative of a generalized function is also a generalized function.

Two generalized functions f and g are called weakly equal if

$$\lim_{k\to\infty} \int_\Omega (f_k(x) - g_k(x))\varphi(x)dx = 0$$

for any representatives f_k and g_k of the classes f and g and for any $\varphi \in C_0^\infty(\Omega)$. Any equation of the form $F(x, u, \dots, D^\alpha u, \dots) = 0, |\alpha| \le m$, where F is a generalized function, can be solved in the sense of weak equality by using an approximation. Of course, the solution is not unique, and the study of its regularity is a rather difficult problem. Some examples of this study can be found in [3].

The authors introduce a new class of generalized functions using the abstract scheme of order completion. They use as the base for their construction the spaces $C^m_{nd}(\Omega)$ of functions f from $C^m(\Omega \setminus \Gamma)$ where Γ is a closed nowhere dense subset of Ω. These spaces are partially ordered by the relation $f \le g$ on $\Omega \setminus \Gamma$. Two functions f and g of this space are equivalent if $f = g$ on $\Omega \setminus \Gamma$ with some Γ. The set of the classes of equivalent functions is an algebra. Its order completion, isomorphic to the set of all cuts, is the space of generalized functions. By the Theorem of MacNeille [4] this set is order complete; that is, each non-empty subset of it has an infimum and a supremum. The space of generalized functions can be identified with the space of all measurable functions because the set of measurable functions is also partially ordered and is order complete.

To solve an equation $T(u) \equiv F(x, u, \dots, D^\alpha u \dots) = 0, |\alpha| \le m$, the authors use the following procedure. For any $\varepsilon > 0$ there exist a closed nowhere dense subset $\Gamma_\varepsilon \subset \Omega$ and a function $u_\varepsilon \in C^m(\Omega \setminus \Gamma_\varepsilon)$ such that $-\varepsilon \le T(u_\varepsilon) \le 0$. Actually Γ_ε is the set of $x \in \Omega$ such that one of the coordinates is equal to $k\delta$ with $k \in \mathbf{Z}$, $\delta > 0$ being fixed, and u_ε is a polynomial in each connected component of $\Omega \setminus \Gamma_\varepsilon$. Let $g \le_T f$ if $T(g) \le T(f)$ on $\Omega \setminus \Gamma$ for some nowhere dense subset Γ. The maximal element u of the set of all u_ε for all $\varepsilon > 0$ with respect to the relation $\le_T$ is by definition a generalized solution to the equation $T(u) = 0$. This solution is unique.

The authors state some applications to specific classes of linear and nonlinear differential equations and study the group invariance of global generalized solutions.

The book is clearly written and can be recommended to specialists in the theory of generalized functions.

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