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Schur algebras and representation theory, by Stuart Martin, Cambridge Tracts in Math., vol. 112, Cambridge University Press, Cambridge, 1993, xv + 232 pp., \$44.95, ISBN 0-521-41591-8

Let V be a vector space of dimension n (over the complex numbers $\mathbb{C}$). The symmetric group $\mathfrak{S}_r$ of permutations of $\{1, 2, \ldots, r\}$ acts on the space $V^{\otimes r}$ of r-tensors by setting $(v_1 \otimes v_2 \otimes \cdots \otimes v_r)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(r)}$, $\sigma \in \mathfrak{S}_r$, $v_1, \ldots, v_r \in V$. As an endomorphism algebra, the (complex) *Schur algebra*

$$S(n, r) = \mathrm{End}_{\mathfrak{S}_r}(V^{\otimes r}), \tag{1}$$

connects the representation theories of $\mathfrak{S}_r$ and the general linear group $GL(n, \mathbb{C})$. To explain how, consider a (matrix) representation $\rho : GL(n, \mathbb{C}) \to GL(m, \mathbb{C})$, $g \mapsto (\rho_{ij}(g))$. We call ρ a *homogeneous polynomial representation of degree* r if each coordinate function ρ_{ij} is a homogeneous polynomial of degree r in the coordinate functions X_{ij}, $1 \leq i, j \leq n$, on $GL(n, \mathbb{C})$. The homogeneous polynomial representations of degree r of $GL(n, \mathbb{C})$ define the objects in a category equivalent to the category $S(n, r)$-mod of finite dimensional modules for $S(n, r)$. From this fact, one can infer information about polynomial representations of general linear groups from the representation theory of symmetric groups. For example, the semisimplicity of the group algebra $\mathbb{C}\mathfrak{S}_r$ implies (by an elementary argument) that the algebra $S(n, r)$ is also semisimple. One can then deduce that any polynomial representation of $GL(n, \mathbb{C})$ is completely reducible.

Definition (1), which stems from 1927 work of Schur [S2], played a central role in Weyl's famous book, *The classical groups* [W]. Actually, Schur had already defined these algebras and used them to study the representation theory of $GL(n, \mathbb{C})$ a full quarter–century earlier in his 1901 dissertation [S1]. Let $A(n) = \mathbb{C}[X_{11}, \ldots, X_{nn}]$ be the polynomial algebra in the n^2 coordinate functions X_{ij} on $GL(n, \mathbb{C})$. Consider the algebra homomorphisms $\Delta : A(n) \to A(n) \otimes A(n)$ and $\epsilon : A(n) \to \mathbb{C}$ defined on the generators X_{ij} by $\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$ and $\epsilon(X_{ij}) = \delta_{ij}$. If $A(n, r)$ denotes the subspace of $A(n)$ consisting of homogeneous polynomials of degree r, then $\Delta(A(n, r)) \subset A(n, r) \otimes A(n, r)$. The linear dual $\Delta^* : A(n, r)^* \otimes A(n, r)^* \to A(n, r)^*$ therefore defines a multiplication on $A(n, r)^*$, which is associative since $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$. Because $(\epsilon \otimes \mathrm{id}) \circ \Delta = (1 \otimes \epsilon) \circ \Delta = \mathrm{id}$, $\epsilon^*(1) \in A(n, r)^*$ is a multiplicative identity. Thus, $A(n, r)^*$ is a unital associative algebra. In fact, it can be proved that

$$S(n, r) \cong A(n, r)^*. \tag{2}$$

When $r \leq n$, Schur showed that there exists an idempotent $e \in S(n, r)$ such that the endomorphism algebra $eS(n, r)e \cong \mathrm{End}_{S(n,r)}(eS(n, r))$ is isomorphic to $\mathbb{C}\mathfrak{S}_r$. By means of this isomorphism, the exact *Schur functor*

$$S(n, r)\text{–mod} \to \mathbb{C}\mathfrak{S}_r\text{–mod}, \quad M \mapsto eM \tag{3}$$

defines an *equivalence* of module categories.

At the time of his dissertation, the complex irreducible characters of the symmetric groups had been worked out by Frobenius [F], who was Schur's advisor. Thus,

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the equivalence (3) provided a vehicle for obtaining precise information about irreducible $S(n,r)$-modules and, equivalently, certain irreducible $GL(n,\mathbb{C})$-modules. In particular, the irreducible $S(n,r)$-modules correspond bijectively to the set $\Lambda^+(r)$ of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of r. When $r \geq n$, the appropriate indexing set is $\Lambda^+(n,r)$, the partitions λ of r into at most n parts. Schur determined the formal characters of the corresponding irreducible modules; these are the now well-known *Schur functions* s_λ, $\lambda \in \Lambda^+(n,r)$, which remain of considerable interest in combinatorics (see [M, p. 32]).

How much of the above theory is characteristic independent, i. e., remains valid when $\mathbb{C}$ is replaced by an infinite field K of positive characteristic p? Definition (1) still makes sense, as does isomorphism (2). The set $\Lambda^+(n,r)$ still indexes the irreducible $S(n,r)$-modules, but the Schur functions no longer give their formal characters. (The s_λ *are* the formal characters of certain $S(n,r)$-modules, namely, the Weyl modules defined by Carter-Lusztig [CL].) The characters of the irreducible modules for both $S(n,r)$ and $K\mathfrak{S}_r$ remain unknown for all p—a problem of considerable interest and intense activity today. Finally, for $r \leq n$, the functor in (3) only realizes $K\mathfrak{S}_r$-mod as a *quotient category* of $S(n,r)$-mod. Observe, for example, that while the number of irreducible $S(n,r)$-modules is characteristic independent, the number of irreducible $K\mathfrak{S}_r$-modules *decreases* if $p \leq r$ to the number of p-regular conjugacy classes.

Green's notes [G] provided an account of the above theory as it stood in 1980. Besides giving a treatment of Schur's work, an account of part of [CL], and a treatment of related work of James [J], Green showed how Schur's original program of going from $\mathfrak{S}_r$ to $GL(n,\mathbb{C})$ by means of the Schur functor (3) might profitably be *reversed* for general K. For example, with this method, Green proved James's theorem that the decomposition matrix of $K\mathfrak{S}_r$ is a submatrix of the decomposition matrix of $GL(n,K)$.

Stuart Martin, the author of the book under review, here states, "My intention in this book is to expand on Green's treatment of Schur algebras, and also to write a fairly full account of the exciting developments which have occurred in the time since Green's work was first published" (p. x). In the first three chapters, he provides a systematic discussion of the basic theory of Schur algebras, largely from the point of view of (2). The discussion includes some of the underlying combinatorics such as the "Straightening Formula" which gives a basis for $A(n,r)$ in terms of standard bideterminants.

It is useful to place the theory of Schur algebras within the larger context of quasi-hereditary algebras (as introduced and studied in [CPS] and [S]). Such algebras arise naturally in Lie theory as well as in related geometric situations (e. g., in the theory of perverse sheaves [PS1], [PS2]). Quasi-hereditary algebras admit recursive constructions, and they have strong homological properties. Martin includes an exposé of the more elementary features of this theory and a proof that the Schur algebras are quasi-hereditary (see also [P] and [PW]).

The introduction of the Schur functor (3) in Chapter 4 serves to draw out the connection with the representation theory of the symmetric groups $\mathfrak{S}_r$. This chapter contains a wealth of information on the modular representation theory of $\mathfrak{S}_r$, including discussions of Specht modules (the images of the dual Weyl modules under the Schur functor), Young modules, and the Mullineux conjecture (now a theorem of Ford and Kleshchev).

Chapter 5 contains a presentation of the theory of blocks for Schur algebras, the high point of which is the discussion of the important "Nakayama's Rule" for $S(n,r)$ (due to Donkin [Do2], [Do3]). In order to keep the exposition a reasonable length, Martin quotes without proof some earlier work of Donkin [Do1] on blocks for semisimple algebraic groups. The final two chapters (6 and 7) focus on other aspects of the theory.

Given $0 \neq q \in K$, the *Hecke algebra* $\mathcal{H}_q$ associated to $\mathfrak{S}_r$ has basis T_σ, $\sigma \in \mathfrak{S}_r$, satisfying the relations:

$$T_{(i,i+1)}T_\sigma = \begin{cases} T_{(i,i+1)\sigma}, & \sigma^{-1}(i) < \sigma^{-1}(i+1) \\ qT_{(i,i+1)\sigma} + (q-1)T_\sigma, & \text{otherwise.} \end{cases} \tag{4}$$

Thus, when $q = 1$, $\mathcal{H}_q \cong K\mathfrak{S}_r$. If $\{e_1, \ldots, e_n\}$ is an ordered basis for V, write $e_J = e_{j_1} \otimes \cdots \otimes e_{j_r}$ for a sequence $J = j_1, \ldots, j_r$ with $1 \leq j_s \leq n$. Then

$$e_J T_{(i,i+1)} = \begin{cases} qe_J(i,i+1), & \text{if } j_i \leq j_{i+1} \\ e_J(i,i+1) + (q-1)e_J, & \text{if } j_i > j_{i+1} \end{cases} \tag{5}$$

defines an $\mathcal{H}_q$-module structure on $V^{\otimes r}$. ($\mathcal{H}_q$-module structures on tensor space were perhaps first considered by Jimbo [Ji, §4].) Of course, when $q = 1$, (5) reduces to the natural action the group algebra $K\mathfrak{S}_r$ considered above. Now define the *q-Schur algebra* $S_q(n,r)$ (over K) to be the endomorphism algebra

$$S_q(n,r) = \mathrm{End}_{\mathcal{H}_q}(V^{\otimes r}). \tag{6}$$

These algebras, first considered by Dipper-James [DJ], represent a natural generalization of Schur algebras. (Sometimes a Morita equivalent version of (6) is used.)

Suppose that K has characteristic $p > 0$ and $q \in \mathbb{Z}$ is a positive power of a prime ℓ *distinct* from p. In work in the 1980s, Dipper and James discovered the remarkable fact that the representation theory over K of the finite general linear group $GL(n, \mathbb{F}_q)$ is closely related to the representation theory of the algebra $S_q(n,r)$ (obtained by replacing q by its image in K). (See [D] for a survey and relevant references.) This connection with the representation theory of finite linear groups in *non-describing characteristic* suggests the importance of developing a theory of q-Schur algebras analogous to the theory of Schur algebras discussed above.

Missing from our discussion so far is a replacement for the group $GL(n,K)$. That replacement turns out not to be a group at all (in any sane person's sense!), but a *quantum group* $GL_q(n,K)$. We can informally think of $GL_q(n,K)$ as defined by its coordinate algebra $\mathcal{O}(GL_q(n,K))$, a non-commutative deformation of the coordinate algebra of $GL(n,K)$. Once $GL_q(n,K)$ has been fixed, the theory of q-Schur algebras can be developed largely parallel to that of Schur algebras. This program has been carried out, for example, in [DD] and [PW]—though the reader should be cautioned that different versions of $GL_q(n,K)$ are used. Fortunately, either choice leads to the same q-Schur algebra provided that $\sqrt{q} \in K$ (a fact which can be nicely explained—see, e. g., [AST] and [DPW]—as part of the theory of multi-parameter quantum groups).

Martin begins (in Chapter 6) by introducing quantum matrix spaces. Then q-Schur algebras are defined and connected with the theory of Hecke algebras. The basic ideas of q-Weyl modules, quantum determinants, and finally quantum general linear groups come to the fore in order to analyze the representation theory of the

q-Schur algebras. The author works with the version of $GL_q(n, K)$ given in [DD]. This choice is largely a matter of taste perhaps, but it has the disadvantage of being less familiar than the standard (Manin) $GL_q(n, K)$. (Manin's $GL_q(n, K)$ has the advantage that the quantum determinant $\det_q$ is a central element in $\mathcal{O}(GL_q(n, K))$.) Chapter 7 concludes with a brief sketch of the Dipper-James results explaining how the theory of q-Schur algebras fits into the representation theory of the finite general linear groups $GL(n, \mathbb{F}_q)$ via the connection between decomposition matrices. Thus, a very appealing picture emerges which connects the representation theory of *modular* quantum groups with that of finite general linear groups in non-describing characteristics—a highly suggestive picture when compared to the well-known work of Steinberg [St] comparing the representation theory of finite groups of Lie type in the describing characteristic with that of the ambient algebraic groups.

Although the material presented in this book is reasonable in its scope, the book does suffer from insufficient editing and a certain lack of attention to detail. For example, the author might profitably have reflected more on his notation; the fact that the symmetric group $\mathfrak{S}_r$ is sometimes denoted Σ_r, sometimes G_r, and sometimes just G makes browsing difficult. Moreover, the informal writing style, while nice in places, hinders when it becomes too informal, as in the long proof of Theorem 2.4.7, where the author confuses the issue of proving a certain set of vectors is a basis by eventually talking about proving its indexing set Ξ is linearly independent. Likewise, the reader will not be comforted by such things as the discussion of " … the ring $\mathcal{S}_{n,r}$ of all symmetric functions of total degree r in the X_i, ..." (p. 19) and by the example on p. 85 apparently asserting (in view of the discussion immediately preceding it) that the ring of integers $\mathbb{Z}$ is a complete discrete valuation ring! These are unfortunate lapses in an otherwise solid treatment of Schur algebras and representation theory.

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