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Mathematical scattering theory: General theory, by D. R. Yafaev, Transl. Math. Monographs, vol. 105, Amer. Math. Soc., Providence, RI, 1992, x + 341 pp., \$216.00, ISBN 0-8218-4558-6

It would be appropriate to begin this review by noting that the book under review is intended to be the first volume of a two-volume set on *Mathematical Scattering Theory* (hereafter abbreviated as MST). The present volume gives, to quote the preface, "a systematic exposition, while oriented toward concrete applications, of the method of abstract scattering theory." Here, abstract may be rephrased as operator-theoretical. To continue the quote, "In the second volume we intend to apply these methods to the theory of differential operators, primarily to the Schrödinger operators." The second volume has not yet appeared. The first volume alone is an independent and complete monograph. Nevertheless, with the second volume to come the author's view of this field will be before us in a more thorough form.

Before proceeding, it should be mentioned that there is another MST, that is, the scattering theory for wave equations originated by P. Lax and R. S. Phillips [LP] and developed extensively since then. Due to its basically hyperbolic nature, it is natural to treat this unique theory separately from MST for Schrödinger operators. The book under review is for MST of Schrödinger type, and we will not comment on MST for wave equations any further.

In order to place this volume in a perspective among the literature on scattering theory, a short comment on the history of MST may be in order. The reviewer apologizes that this has some inevitable overlap with the introduction of this volume, in which the author presents his version together with a motivation of MST in a rough explanation of the contents of the volume.

The scattering theory has its origin in physics, in particular scattering phenomena in quantum mechanics. Its central object, the scattering operator, relates the past asymptotic state of a dynamical system governed by the Schrödinger equation to its future asymptotic state. To be specific, the Schrödinger equation is given by $i\dfrac{d}{dt}u(t) = Hu(t)$. Here H is the Hamiltonian of the system and, according to von Neumann's theory, H is a selfadjoint operator in a Hilbert space. The associated dynamics is described by the unitary group $U(t) = \exp(-itH)$, $t \in \mathbf{R}$. In the scattering theory the dynamics $U(t)$ is compared, when $t \to \pm\infty$, with asymptotic dynamics, which is denoted by $U_0(t)$. In a simple perturbation-theoretic approach the Hamiltonian H has the form $H = H_0 + V$, where H_0 is the free Hamiltonian, and the asymptotic dynamics is the free dynamics $U_0(t) = \exp(-itH_0)$. The perturbation V is supposed to be small relative to H_0, quantitatively or qualitatively. A typical and most important H is the Schrödinger operator

$$H_0 = -\triangle, \quad H = -\triangle + V, \qquad \text{both acting in} \quad L^2(\mathbf{R^3}).$$

Here, V is the operator of the multiplication by a real-valued function $V(x)$ which is called a potential. It is expected that the scattering occurs if $V(x)$ decays at infinity sufficiently rapidly.

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Asymptotic states $u_\pm$ of $U(t)u$ are characterized by the relation

$$U(t)u \sim U_0(t)u_\pm, \qquad t \to \pm\infty. \tag{1}$$

The *wave operator*, which maps asymptotic states to the original state, is defined by

$$W_\pm = W_\pm(H_1, H_0) = \underset{t\to\pm\infty}{s\text{-}\lim}\, U_1(-t)U_0(t)P_0, \tag{2}$$

and the *scattering operator*, which maps the past asymptotic state to the future one, by

$$S = S(H_1, H_0) = W_+(H_1, H_0)^* W_-(H_1, H_0).$$

The additional factor P_0 in (2) is added in order to project out eigenspaces of H_0. In an early stage of MST it was discovered that the correct P_0 is the projection onto the space of absolute continuity of H_0. (Except for rare events that H and H_0 have common eigenvectors with the same eigenvalue, (1) with $U_0(t) = \exp(-itH_0)$ requests that u and $u_\pm$ are orthogonal to eigenspaces of H and H_0, respectively. So the scattering theory is a phenomenon associated with continuous spectra.)

The wave operator is a main object of research in MST. $W_\pm$ is isometric on the range of P_0 whenever the strong limit in (2) exists. Let us denote the range of $W_\pm$ by $\mathcal{R}_\pm$. If $\mathcal{R}_+ = \mathcal{R}_-$, then S is unitary. The relation $\mathcal{R}_+ = \mathcal{R}_-$, or a stronger requirement that $\mathcal{R}_\pm$ are both equal to the orthogonal complements of the span of eigenspaces of H, is called the *completeness* of wave operators. The completeness in the stronger sense implies that H has no singular continuous spectrum and that the absolutely continuous part of H is unitarily equivalent to that of H_0. At the first encounter the wave and the scattering operators may look like rather peculiar mathematical objects. In the course of development of MST, on the contrary, the problem of the existence and the completeness of wave operators has led the development of MST and thus opened up the mathematical spectral theory of Schrödinger operators, which in turn has shown itself to have a surprisingly rich structure.

MST has two faces. One may be called *abstract scattering theory*. Here, MST is a branch of functional analysis or operator theory and finds a strong tie with spectral theory, especially with the perturbation theory of continuous spectra through the aforementioned unitary equivalence of absolutely continuous parts. The other face may be called the *scattering theory of Schrödinger equations*. Here abundant mathematical features possessed by Schrödinger equations, not necessarily directly related to the scattering, have been targets of persistent and intensive study. The equation ranges from that of a simple potential scattering to that of highly complicated systems like N-particle systems.

Concerning the method there are two main approaches in MST. One is called the *time-dependent method*, and the other the *stationary method*. In a time-dependent approach one deals with the wave operator itself or the asymptotic behavior of $U(t) = \exp(-itH)$ more or less directly. In a stationary approach, which is formally the Laplace transform of a time-dependent formulation, one deals with boundary behavior of the resolvent $R(z, H) = (H - z)^{-1}$ in a suitable topology. This approach for partial differential operators is sometimes highlighted by the name "limiting absorption principle". This somehow corresponds to the Lippmann-Schwinger equation famous in quantum mechanical scattering theory.

A potential V is called short-range if $V(x) = O(|x|^{-\delta})$, $|x| \to \infty$, with $\delta > 1$. Until around 1970, when the completeness problem was settled for short-range potentials, the abstract scattering theory and the scattering theory for Schrödinger equations went hand in hand. Except for problems of eigenfunction expansions the general trend was that criteria obtained in abstract scattering theory are applied to Schrödinger equations with the aid of some estimates. As the research of Schrödinger operators proceeded to problems with long-range potentials (i.e., with $0 < \delta \leq 1$) and to problems of N-body systems, main features of the problem can no longer be formulated in general operator-theoretical terms. One example is the use of modified free dynamics $U_0(t) = \exp(\phi_t(H_0))$ in a long-range potential scattering, where how to choose ϕ_t is inherent in the system under consideration. Around the same time two new methods came on the scene. One is the time-dependent approach by V. Enss [E]. This is a theory for Schrödinger equations and revealed the effectiveness of studying propagation behavior associated with $\exp(-itH)$. (In fact, the claim was then around that it was the first physically meaningful proof of the completeness!) The other is an abstractly formulated commutator method, often called the Mourre estimate [M], which since then has provided powerful means for various problems of Schrödinger operators. The method combines, somewhat implicitly, compactness criteria with the commutation relation $[P, Q] = -iI$ characteristic to Schrödinger dynamics. Thus, the 1970's being a transient period, the swing of the pendulum has been on the side of PDE methods and methods of asymptotic analysis, such as estimates of oscillatory integrals, combined with Enss and Mourre approaches.

As mentioned at the beginning, the volume under review narrates solely abstract scattering theory. After a general formulation of MST is given in Chapter 2, which is of course in time-dependent terms, the emphasis is laid on the side of stationary methods. Abstract approaches, as explained above, reached more or less their present-day form in the early 70's. In this respect this first volume treats roughly the first half of the story. (A notable exception is the latter part of Chapter 7.) Nevertheless, it does not diminish the value of this volume, even viewed as an independent book. In fact, though the theory was formed up to the 70's, a systematic and thorough treatise on abstract scattering theory is rather scarce. [K] contains the theory up to the mid 60's, and [RS] has much of abstract scattering theory, but scattered in four volumes. There are several books on scattering theory which lean more to treating Schrödinger operators (e.g., [P], [AJS]). These books contain some of the abstract methods. Perhaps [BW] is the only single-volume treatise which contains a thorough and systematic account of the abstract scattering theory. Comparison by the author himself with other books is found in the preface. We only mention that [P] does not appear in the bibliography of this volume.

Abstract methods, though they may not be the most powerful in each concrete problem of today, should retain the merit of being universal. For further development of the method and possible future use, it is hoped that a single volume will present abstract scattering theory with all its perspectives. The volume under review serves this purpose well. It is particularly welcome that the author presents (1) a general abstract stationary method from his viewpoint and (2) an elegant story of the so-called trace class theory, including a full account of Krein's spectral shift function and trace formula. The book also serves as a systematic introduction to MST and is certainly a welcome addition to the literature.

The book consists of eight chapters. Chapter 1 is concerned with preliminary facts which are beyond a standard course on functional analysis and contains concise but nice sections on "classes of compact operators", "the trace and determinant", and "the analytic Fredholm theory".

The basic part of the theory of wave and scattering operators is presented in Chapters 2 and 3. Chapter 2 also contains a preview of the general stationary approach.

Chapter 4, "Scattering for relatively smooth perturbations", deals with two topics, the Friedrichs-Faddeev model and the Kato smoothness. The former is a classical model from which MST started. There, the perturbation is expressed by a Hölder continuous integral kernel in the spectral representation space. The Kato smoothness is the first and yet close-to-final smoothness condition in abstract terms. One big virtue of the smoothness theory, apart from its applicability, is that the bridge between the time-dependent and the stationary formulation is comfortably straightforward. The Friedrichs-Faddeev model has not yet been discussed so completely in a monograph, and it is good to see it here.

The topic of Chapter 5, "The general scheme in stationary scattering theory", is the so-called abstract (or axiomatic) stationary method. The aim is to construct an abstract theory (or rather a framework) pinpointing essential features commonly present in various situations, from the smoothness situation to the trace class perturbation where, in the latter, no kind of smoothness can be expected in general. There are a few versions of the abstract stationary method, among which one is by T. Kato and the reviewer [KK]. Here, a systematic presentation is given along the lines of [BY], which deals with the scattering theory with two Hilbert spaces. Inasmuch as this chapter is a high point in this monograph, the reviewer would have to add that the chapter is not the easiest to read. Perhaps this is partly because the author explains the theory first on a formal level in §7 of Chapter 2 and then gives rigorous exposition or justification in Chapter 5. This is good in giving the reader a beforehand perspective, but in reading Chapter 5 one has to go back and forth, which I found a bit annoying.

Chapter 6, "Scattering for perturbations of trace class type", deals with the theory that was the first milestone in the development of the abstract scattering theory. Here, possibly for the first time in a book, the trace class perturbation theory is presented as an application of the abstract stationary method of Chapter 5. Later in the chapter, a time-dependent proof of E. B. Pearson (1978) is given.

Up to Chapter 6 most of the standard material of abstract scattering theory is covered, with the author's own view here and there. Then come two more chapters unique to this volume. Chapter 7, "Properties of the scattering matrix (SM)", is devoted to a thorough discussion of the scattering matrix $S(\lambda)$, a representation of the scattering operator on an energy shell. Starting from a widely used representation formula for the scattering matrix, the author examines detailed properties of $S(\lambda)$, culminating in the proof of a pointwise bound of $|S(\lambda)-I|_p$ [SY], where $|\quad|_p$ is the norm of the operator class S_p in the Hilbert space associated with the energy shell. It is good to see it here as a token of the recent role of abstract stationary theory.

Chapter 8, "The spectral shift function (SSF) and the trace formula", is devoted to the trace formula of M. G. Krein and the Birman-Krein formula on the connection of the spectral shift function and the scattering matrix. This much-discussed theory, nevertheless, has never been presented in a monograph in this unified form. This chapter will become, together with an expository article [BY2], a standard reference on the subject. I would say that the last three chapters form another highlight of this volume. In particular, the expositions in these chapters are the most lucid in this volume.

"Review of the Literature" at the end of the volume is conveniently of moderate size. These reviews and the bibliography naturally put some emphasis on Russian works, in particular, the St. Petersburg School. But this I found useful rather than unbalanced. Incidentally, [BW] is cited on page 329 with an incorrect reference number ([3] instead of [30]). I mention this because of the multi-alphabetical numbering of the book section of the bibliography.

Some adverse comments are due. The first is that, perhaps with the coming second volume in sight, the author gave no applications to Schrödinger operators in this volume, even at the level of examples. Inasmuch as the Schrödinger operator appears in the "Introduction" as a strong motivation to scattering theory, it is regrettable that a beginning reader could finish the volume without learning any concrete result on Schrödinger operators. The second comment is that the author tends to present materials in a general way, i.e., with the least assumptions possible. Occasionally, this makes the reading somewhat hard. Due to these problems a reader of this book might perhaps have to be already well motivated to learn scattering theory seriously. There are some, but not very many, misprints and minor errors, which are easily spotted and corrected. I only mention that on two occasions (the 4th line of page 75 and the first formula in Lemma 2 on page 251) $\cap$ should be $\cup$.

Except for the comments above I found that the volume is well organized and is a fine guide for a serious reader and a good source of information to research workers in the field. And I look forward eagerly to the publication of the second volume.

The final comment is on the price. One of my friends reminded me that the book costs $216 for 341 pages and called attention to the fact that it makes it all but impossible to assign this volume as a textbook in a course. This price, even though it is $130 for AMS members, seems exceptionally dear. The book is from the AMS, and we hope that some thought would be given to this difficulty.

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