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An introduction to homological algebra, by Charles A. Weibel, Cambridge Studies in Advanced Math., vol. 38, Cambridge Univ. Press, 1994, xiv + 450 pp., $24.95, ISBN 0-521-55987-1

When I was a graduate student at the University of Chicago, homological algebra was an unpopular subject. The general attitude was that it was a grotesque formalism, boring to learn, and not very useful once one had learned it. Perhaps an algebraic topologist was forced to know this stuff, but surely no one else should waste time on it. The few true believers were viewed as workers at the fringe of mathematics who kept tinkering with their elaborate machine, smoothing out rough patches here and there.

This attitude changed dramatically in 1958 when Serre characterized regular local rings using homological algebra (they are the (commutative) local rings of finite global dimension) and then used this criterion to prove that any localization of a regular local ring is itself regular (until then, only special cases of this were known). About the same time, Auslander and Buchsbaum completed work of Nagata by using global dimension to prove that every regular local ring is a unique factorization domain.

In spite of its newfound popularity, homological algebra still "got no respect". For example, the two theorems above used the notion of homological dimension of a module. In 1958 Kaplansky reluctantly decided to teach homological algebra. One of his students, Schanuel, noticed that there was an elegant relation between different projective resolutions of the same module. Kaplansky seized this result, nowadays called Schanuel's Lemma, for it allowed one to prove both the theorems of Serre and Auslander-Buchsbaum without having first to develop Ext or Tor (this account can be found in Kaplansky's mimeographed notes from that time, later published in his *Commutative rings*, 1970).

As more applications were found and as more homology and cohomology theories were invented to solve outstanding problems, resistance to homological algebra waned. Indeed, today it is just another standard tool in a graduate student's mathematical kit. On the other hand, pedagogical problems remain in presenting it because, at the outset, it is still rather heavy on definitions and light on applications.

The basic idea of homology comes from Green's theorem, where a double integral over a region R with holes is equal to a line integral on the boundary of R. Poincaré recognized that whether a topological space X has different types of "holes" is a kind of connectivity. To illustrate, let us assume that X can be triangulated, so it can be partitioned into points, edges, triangles, tetrahedra, and higher-dimensional analogues, called 0-simplexes, 1-simplexes, 2-simplexes, 3-simplexes, etc. The question to ask is whether a union of n-simplexes in X that "ought" to be the boundary of some $(n+1)$-simplex actually is such a boundary. For example, when $n = 0$, two points a and b in X ought to be the boundary of an edge in X; if there is no path in X joining a and b, then there is a 0-dimensional hole in X (we say that X is *path connected* if it has no such holes). For an example of a 1-dimensional hole, let X be the plane with the origin deleted. The perimeter of a triangle ought to be

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the boundary of a 2-simplex—its interior, but this is not so if the triangle contains the origin in its interior; thus, X has a 1-dimensional hole. Such ideas led to the following construction. For each $n \geq 0$, consider all formal linear combinations of n-simplexes (union is replaced by sum, and simplexes are "oriented" to mirror the data of Green's theorem; this is the reason negative coefficients are allowed); call such linear combinations *n-chains*. Certain of these n-chains "ought" to be boundaries of some union of $(n+1)$-simplexes; call them *n-cycles* (for example, adding up the edges of a triangle, with appropriate choice of signs, is a 1-cycle). And certain n-cycles actually are boundaries, and they are called *n-boundaries*. All the n-chains in X form an abelian group under addition, denoted by $C_n(X)$; all the n-cycles form a subgroup, denoted by $Z_n(X)$; and all the n-boundaries form a subgroup of $Z_n(X)$, denoted by $B_n(X)$. The interesting construct from this is the quotient group $Z_n(x)/B_n(X)$, which is called the *nth homology group* $H_n(X)$. What survives in this quotient are the n-dimensional holes, and so these groups measure more subtle kinds of connectedness. For example, $H_0(X) = 0$ means that X is pathwise connected. Homological algebra really arose in trying to compute and to find relations among these abelian groups. A key ingredient in the construction of homology groups is that the subgroups Z_n and B_n can be defined via homomorphisms; there are *boundary homomorphisms* $\partial_n : C_n(X) \to C_{n-1}(X)$ with $\ker \partial_n = Z_n(X)$ and im $\partial_{n+1} = B_n(X)$, and so there is a sequence of abelian groups and homomorphisms

$$\cdots \to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X).$$

An analogous construction arose in invariant theory a century ago. One knows that every group G has a *presentation*; that is, it can be described by *generators* $x_1, x_2, \ldots$ and *relations* (certain expressions $r_1, r_2, \ldots$ in the x_i's are 1). This idea was made precise in the 1880s by von Dyck, who defined free groups. A group is *free* if it is generated by a *basis*, that is, a subset behaving, with respect to constructing homomorphisms, just as a basis of a vector space does (there exists a unique linear transformation taking any specified value on each of its basis elements). If F is the free group on the x_i's and if R is the (normal) subgroup of F generated by the r_j's, then $G \cong F/R$. Similarly, an ideal I in the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$ can almost be described by generators and relations. Define an R-module to be free if it has a basis. If I is generated by $a_1, \ldots, a_t$ and if F is a free module with basis $A_1, \ldots, A_t$, then the ideal I is a homomorphic image of F; there is a surjective homomorphism $\varphi : F \to I$, defined by $\varphi(A_i) = a_i$ for all i, and hence $I \cong F/\ker\varphi$. If $\ker\varphi$ has a basis, that is, if $\ker\varphi$ is free, then there is a presentation of I with generators and relations. But $\ker\varphi$ need not be free. One now iterates this construction: Take a homomorphism $\varphi_2 : F_2 \to \ker\varphi$, where F_2 is a free module; that is, we seek "relations on the relations" (called *syzygies*). If $\ker\varphi_2$ is not free, map a free module onto it. Hilbert's theorem on syzygies says that if this is done n times, where R is the polynomial ring in n variables, then $\ker\varphi_n$ is guaranteed to be free. The sequence $F, F_2, F_3, \ldots$ of free modules so obtained can be assembled into an exact sequence

$$\cdots \to F_4 \to F_3 \to F_2 \to F \to I \to 0,$$

where the arrows are essentially the homomorphisms φ_i (*exact* sequence means that the image of every arrow equals the kernel of the next arrow).

What is needed to define homology is a sequence as above, but it need not be exact, nor must its terms be free; one demands only that the composite of adjacent arrows be 0. Such do arise, for example, by applying functors such as Hom and tensor to exact sequences. In this way, one gets the groups Ext and Tor.

In the 1940s and 1950s, it was recognized that theorems in different fields led to similar constructions. For example, there was Hilbert's Theorem 90 about algebras, Whitehead's lemmas about Lie algebras, and Schreier's solution of the extension problem in group theory. Sheaves arose in several complex variables and in algebraic geometry. Each of these structures has a sequence of homology groups associated to it, and many older theorems could be interpreted in this new way.

Homology is more than a new language that organizes many computations into a coherent theory. It surely is such a language, and one should not sneer at this, for such a language allows one to distinguish the routine from the difficult, allowing one to focus on essentials. The self-deprecating phrase *general abstract nonsense* (due to Steenrod) was promulgated by Eilenberg and Mac Lane, two of the major innovators of homological algebra, to highlight this aspect of the subject. Along with homology groups, however, there are methods to compute them, and this is what makes the decisive contribution. Although one can calculate many things without them, the most powerful method of computing homology groups uses spectral sequences. When I was a graduate student, I always wanted to be able to say, nonchalantly, that such and such is true "by the usual spectral sequence argument," but I never had the nerve.

It is now appropriate to quote from the book under review.

> Until 1970, almost every mathematician learned the subject from Cartan-Eilenberg. The canonical list of subjects (Ext, Tor, etc.) came from this book. As the subject gained in popularity, other books gradually appeared on the subject: Mac Lane's 1963 book, Hilton and Stammbach's 1971 book, Rotman's 1970 notes later expanded into his 1979 book, and Bourbaki's 1980 monograph come to mind. All these books covered the canonical list of subjects, but each had its own special emphasis.
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> In the meantime, homological algebra continued to evolve. In the period 1955–1975 (sic), the subject received another major impetus, borrowing topological ideas. The Dold-Kan correspondence allowed the introduction of simplicial methods, $\lim^1$ appeared in the cohomology of classifying spaces, spectral sequences assumed a central role in calculations, sheaf cohomology became part of the foundations of algebraic geometry, and the derived category emerged as the formal analogue of the topologists' homotopy category.
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> Largely due to the influence of Grothendieck, homological algebra became increasingly dependent on the central notions of abelian category and derived functors. The cohomology of sheaves, the Grothendieck spectral sequence, local cohomology, and the derived category all owe their existence to these notions. Other topics, such as Galois cohomology, were profoundly influenced.
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> Unfortunately, many of these later developments are not easily found by students needing homological algebra as a tool. The effect is a technological barrier between casual users and experts at homological algebra.

> This book is an attempt to break down that barrier by providing an introduction to homological algebra as it exists today.

Weibel's book consists of ten chapters followed by a brief appendix entitled "Category Theory Language" (with sections "Categories", "Functors", "Natural Transformations", "Abelian Categories", "Limits and Colimits", and "Adjoint Functors").

The first seven chapters treat "old" material from the pre-1980 era. Chapter titles are: "Chain Complexes", "Derived Functors", "Tor and Ext", "Homological Dimension", "Spectral Sequences", "Group Homology and Cohomology", "Lie Algebra Homology and Cohomology".

The last three chapters, roughly 40 percent of the text, deliver the more recent goods, and so it is worth mentioning their content in more detail. "Simplicial Methods in Homological Algebra": simplicial objects; operations on simplicial objects; simplicial homotopy groups; Dold-Kan correspondence; Eilenberg-Zilber theorem; canonical resolutions; cotriple homology; André-Quillen homology and cohomology. "Hochschild and Cyclic Homology": Hochschild homology and cohomology of algebras; derivations, differentials, and separable algebras; H^2, extensions, and smooth algebras; Hochschild products; Morita invariance; cyclic homology; group rings; mixed complexes; graded algebras; Lie algebras of matrices. "The Derived Category": triangulated categories; localization and the calculus of fractions; derived category; total tensor product; Ext and $\mathbf{R}\,\mathrm{Hom}$; replacing spectral sequences; topological derived category.

Although the first seven chapters cover well-trodden paths, there are interesting examples all along the way. (I do feel that Auslander and Buchsbaum, who are explicitly credited with their codimension theorem, are overlooked when their theorem asserting unique factorization in regular local rings is stated.) As Weibel says in his introduction, there is a need for an exposition of the recent developments. However, the pedagogical problem I mentioned at the outset of this review persists. My feeling is that most students first learning homological algebra would find this book difficult to digest. On the other hand, for anyone who has already seen some homological algebra, or for anyone who has a particular problem in mind that might benefit from homological methods, I do recommend this book. By collecting, organizing, and presenting both the old and the new in homological algebra, Weibel has performed a valuable service. He has written a book that I am happy to have in my library.

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