RESEARCH REPORTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 4, October 1996

TIGHT CLOSURE

WINFRIED BRUNS

Abstract. The theory of tight closure was created by Mel Hochster and Craig Huneke about ten years ago. Assisted by numerous contributions of others, they have continuously developed the theory since then. 'Tight closure' can now be regarded as a synonym for 'characteristic p methods in commutative algebra'. It ties several strands of commutative algebra and algebraic geometry together: invariant theory, rational singularities, the Briançon–Skoda theorem, the 'homological conjectures', big Cohen–Macaulay modules and algebras, and various other topics.

The theory of tight closure was created by Mel Hochster and Craig Huneke about ten years ago. They presented it in several conferences in the spring of 1987 and especially in a workshop at the MSRI, Berkeley, of which Huneke's *Intelligencer* article [Hu1] is a lively account. Assisted by numerous contributions of others, they have continuously developed the theory since then (see [HH2], [HH3], [HH4], [HH5], [HH6], [AHH]), and 'tight closure' can now be regarded as a synonym for 'characteristic p methods in commutative algebra'. It ties several strands of commutative algebra and algebraic geometry together: invariant theory, rational singularities, the Briançon–Skoda theorem, the 'homological conjectures', big Cohen–Macaulay modules and algebras, and various other topics.

The choice of material for this report is certainly a subjective one; nevertheless I think that the most important theorems on tight closure itself and those proved by its application have been included. Although a research report is not the place for systematic development, I hope that the main ideas will become visible and that the brevity of exposition has not buried the perhaps most striking feature of tight closure, namely, the elegance with which it yields proofs of theorems that are very hard to get at by more traditional methods.

Received by the editors March 21, 1996, and, in revised form, May 27, 1996.

1991 *Mathematics Subject Classification.* Primary 13A35, 13A50, 13B21, 13D25, 13H10, 14B05.

Key words and phrases. Tight closure, characteristic p methods, invariant theory, rational singularities, Briançon–Skoda theorem, homological conjectures, phantom homology.

I am very grateful to Craig Huneke for providing me with a preprint version of his lecture notes [Hu3], which I highly recommend as an introduction to tight closure and without which writing this report would have been much more laborious.

We shall use the terminology of commutative algebra as developed in Bruns and Herzog [BH].

1. The Frobenius functor

When we say a ring R has characteristic p, then it is tacitly understood that $p > 0$ is a prime number and $pa = 0$ for all $a \in R$. In other words: R contains the field $\mathbb{Z}/p\mathbb{Z}$. A peculiar feature of characteristic p is the *Frobenius map* $F\colon R \to R$, $F(a) = a^p$. Since $(a+b)^p = a^p + b^p$, F is an endomorphism of R. For the application to modules one must extend F to the *Frobenius functor* $\mathcal{F}$, using a second copy R^F of R which is regarded as an R-module via the homomorphism $F\colon R \to R^F$. Set $\mathcal{F}(M) = M \otimes_R R^F$ for each R-module M, and view $\mathcal{F}(M)$ as an R-module via the *identification* $R = R^F$. It follows easily that $\mathcal{F}$ is additive, $\mathcal{F}(R) = R$, and $\mathcal{F}(a\varphi) = a^p\mathcal{F}(\varphi)$ for $a \in R$ and an R-linear map φ.

The 'philosophy' of applying the Frobenius map can be described as follows. Let $\mathcal{S}$ be a set of polynomial equations $f(x) = 0$, $f \in R[X_1, \dots, X_n]$. If $x \in R^n$ is a solution of $\mathcal{S}$, then $F(x) = (F(x_1), \dots, F(x_n))$ is a solution of $F(\mathcal{S})$ where $F(\mathcal{S})$ arises from $\mathcal{S}$ by applying F to the *coefficients* of the polynomials f so that, in a sense, $F(\mathcal{S})$ has the same type as $\mathcal{S}$. If R is local with maximal ideal $\mathfrak{m}$ and $x \in \mathfrak{m}R^n$, then the solutions $F^e(x)$ of the systems $F^e(\mathcal{S})$ converge to zero in the $\mathfrak{m}$-adic topology when e tends to ∞. If, on the other hand, one can bound such solutions 'below', then a contradiction to the solubility of $\mathcal{S}$ has been achieved.

In a very important case we not only get solutions of $F(\mathcal{S})$ from those of $\mathcal{S}$, but we also know all the solutions of $F(\mathcal{S})$.

Theorem 1.1 (Peskine–Szpiro)**.** *Let*

$$\mathbb{F}\colon 0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be an acyclic complex of finitely generated free R-modules. Then $\mathcal{F}(\mathbb{F})$ is also acyclic.

The complex $\mathcal{F}(\mathbb{F})$ has a very plain description: the free modules do not change, and for each i one replaces the matrix representing φ_i by the matrix of the p-th powers of its entries. Theorem 1.1 is a key result of Peskine and Szpiro's pioneering work [PS] that paved the way for the application of characteristic p methods to problems in homological algebra.

Over a regular local ring R every finite R-module has a finite free resolution $\mathbb{F}$ as in the theorem, and therefore regular rings are distinguished with respect to the Frobenius map:

Theorem 1.2 (Kunz [Ku])**.** *Let R be a Noetherian ring of characteristic p. Then $\mathcal{F}$ is an exact functor if and only if R is regular.*

Recall that R is regular if its localizations $R_\mathfrak{m}$ are regular local rings (this convention also applies to other ring-theoretic properties). Therefore the implication $\Leftarrow$ follows immediately from 1.1. Since $\mathcal{F}$ is exact if and only if R is a flat R-module via the homomorphism $F\colon R \to R$, one simply speaks of the *flatness of the Frobenius* (map).

In the context of tight closure one must distinguish three classes of rings: (i) those of characteristic p, (ii) those containing a field of characteristic 0, and (iii) those not containing a field. A local ring R with maximal ideal $\mathfrak{m}$ belongs to one of the classes (i) or (ii) if it is equicharacteristic, i.e. R and $R/\mathfrak{m}$ have the same characteristic; otherwise it is said to have mixed characteristic.

Many theorems proved by using the Frobenius map can be shown to hold for rings containing a field of characteristic 0 as well. They are derived by *reduction to characteristic p*. This is a delicate procedure and most often requires the use of Artin's approximation theorem. We refer the reader to Hochster [Ho1] or [BH, Ch. 8] for a demonstration.

2. Tight closure

Let I be an ideal in R. Then $\mathcal{F}(R/I) = R/F(I)R$, and $F(I)R$ is the ideal generated by the p-th powers of the elements (or just of a system of generators) of I. An e-fold iteration of this process leads us to the *Frobenius power* $I^{[q]}$ with $q = p^e$, which is generated by the q-th powers of the elements of I. (In the sequel q will always denote a power of p, and the phrase 'for all $q = p^e$ with $e \gg 0$' will be replaced by the unprecise but shorter 'for all $q \gg 0$'.) Furthermore let R° denote the set of elements of a ring R that are not contained in a minimal prime ideal.

Definition 2.1. Let R be a ring of characteristic p and I an ideal of R. Then the *tight closure* I^* of I is the set of elements x in R such that there exists $c \in R^\circ$ with $cx^q \in I^{[q]}$ for all $q \gg 0$. An ideal I with $I = I^*$ is *tightly closed*.

Obviously I^* is an ideal, and tight closure has the properties usually required for a closure operation, i.e. $I \subset I^*$, $(I^*)^* = I^*$, and $I \subset J \implies I^* \subset J^*$. Furthermore $x \in I^*$ if and only if $x \in ((I + \mathfrak{p})/\mathfrak{p})^*$ for all minimal prime ideals $\mathfrak{p}$; this property reduces many questions to the case in which R is an integral domain. Another simple but important rule is that $I : J$ is tightly closed for every ideal J if I is tightly closed. Here $I : J = \{x \in R\colon xJ \subset I\}$, and using this notation we can rephrase the definition as saying that $c \in I^{[q]} : (x)^{[q]}$ for $q \gg 0$.

The definition of tight closure uses the 'uniform annihilator' c. Such uniform annihilators appear in several of the developments preceding tight closure; for example see Roberts's proof [Ro], [BH, 8.2.6] of the 'new intersection theorem'.

A crucial property of tight closure is given by the next theorem, which also shows that the attribute *tight* is well chosen.

Theorem 2.2 ([HH2]). *Let R be a regular ring of characteristic p. Then every ideal I of R is tightly closed.*

It is enough to show 2.2 for regular local rings $(R, \mathfrak{m})$, and if $x \notin I$, then $I^{[q]} : (x^q) = (I : (x))^{[q]} \subset \mathfrak{m}^q$ where the equation follows from the flatness of the Frobenius map (applied to a suitable exact sequence): the uniform annihilator of $x^q \mod I^{[q]}$, $q \gg 0$, belongs to $\bigcap \mathfrak{m}^q = 0$; in other words, $x \notin I^*$.

The transfer of ring-theoretic properties via a homomorphism $R \to S$ depends to a large extent on what happens to an ideal I of R if we first extend it to S and then contract the extension back to R. In general one cannot expect $IS \cap R = I$, but in many situations the contraction of IS is contained in the tight closure of I. A first example:

Theorem 2.3 ([HH2]). *Let R be a Noetherian integral domain of characteristic p and $S \supset R$ a module-finite extension ring of R. Then $(IS)^* \cap R \subset I^*$ for all ideals I of R.*

In 2.3 S can be replaced by any ring that is a union of module-finite extensions of R. In particular one can replace it by a ring that is the union of 'all' such extensions, namely, the integral closure R^+ of R in an algebraic closure of its field of fractions. It would simplify the theory significantly if one could replace tight closure by R^+-closure. While it is not known whether $IR^+ \cap R = I^*$ in general, this equality holds in an important special case. (Recall that a system of parameters of a Noetherian local ring R of dimension d is a sequence $x_1, \dots, x_d$ of elements of R generating an ideal whose radical is the maximal ideal of R.)

Theorem 2.4 (Smith [Sm2]). *Let R be an excellent local domain of characteristic p and I an ideal generated by a subset of a system of parameters. Then $IR^+ \cap R = I^*$.*

A fundamental result of Hochster and Huneke, closely related with tight closure theory, says that R^+ is a big Cohen–Macaulay algebra if R is an excellent domain of characteristic p; see [HH7].

Tight closure is a powerful but also a complex notion. Its complexity is best indicated by the failure of all attempts to prove that it commutes with localization: one obviously has $(I^*)_{\mathfrak{p}} \subset (I_{\mathfrak{p}})^*$, but the converse inclusion is not known. The largest class of ideals for which localization is known is given by the ideals of finite phantom projective dimension [AHH]; finite phantom projective dimension will be discussed in Section 6. Furthermore, no algorithm for the computation of I^* has been found so far.

The difficulty results from the fact that the assertion $x \in I^*$ comprises infinitely many conditions and that, furthermore, the element c may depend on I and x. At least the second complication can often be avoided. An element c of R° such that $cx^q \in I^{[q]}$ for all x, I, and $q \gg 0$ is called a *test element*.

Theorem 2.5. *Suppose that the reduced ring R is F-finite, i.e. a finite module over itself with respect to the Frobenius map, and let $c \in R$ such that $R_{\mathfrak{p}}$ is a regular local ring for all prime ideals $\mathfrak{p}$ of R not containing c. Then a power of c is a test element.*

Using test elements, one can prove the *persistence of tight closure*.

Theorem 2.6. *Let R be as in* 2.5, *S be a Noetherian ring, and $\varphi\colon R \to S$ a homomorphism. Then $\varphi(I^*) \subset (\varphi(I)S)^*$ for all ideals I of R.*

For the sake of simplicity we have not reproduced the most general versions of 2.5 and 2.6; see [HH5, 6.2 and 6.24]. The condition of F-finiteness is not very restrictive; it is satisfied by all affine and analytic algebras over perfect coefficient fields.

Tight closure can also be defined in characteristic 0 by reduction to characteristic p. We refer the reader to Hochster's appendix of [Hu3] for this technically more demanding notion. So far there is no definition of tight closure in mixed characteristic.

3. Direct summands of regular rings

One is tempted to read Theorem 2.2 as saying that tight closure is an ideal-theoretic invariant measuring the flatness of the Frobenius. Fortunately this interpretation is not precise enough: if the converse of 2.2 were true, then tight closure would lose its impact. It is rather a concept—and in fact the best one available—that conveys the consequences of the flatness of the Frobenius to non-regular rings. This role is very clearly demonstrated by the next two theorems. In the first we descend from the regular ring to the non-regular one.

Theorem 3.1 ([HH2]). *Let S be a regular integral domain of characteristic p or, more generally, a Noetherian ring in which every ideal is tightly closed. If the subring $R \subset S$ is a direct summand of S as an R-module, then every ideal of R is tightly closed.*

In fact, the direct summand property implies that $IS \cap R = I$ for all ideals I of R, and this weaker property already suffices: $I^* \subset I^*S \cap R \subset (IS)^* \cap R = IS \cap R = I$ because of 2.2.

A central notion of the homological and combinatorial aspects of commutative ring theory is that of a Cohen–Macaulay ring. Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. The elements $x_1, \dots, x_k \in \mathfrak{m}$ form a regular sequence if x_{i+1} is not a zero-divisor modulo the ideal $(x_1, \dots, x_i)$ for $i = 1, \dots, k-1$, in other words, if $(x_1, \dots, x_i) : x_{i+1} = (x_1, \dots, x_i)$ for $i = 0, \dots, k-1$. One calls R *Cohen–Macaulay* if one (equivalently, every) system of parameters $x_1, \dots, x_d$ is a regular sequence. Roughly speaking, the next theorem therefore says that every local domain in characteristic p is Cohen–Macaulay *up to tight closure*.

Theorem 3.2 ([HH2]). *Let R be a local domain of characteristic p that is a homomorphic image of a Cohen–Macaulay ring. Then for every system of parameters $x_1, \dots, x_d$ of R one has*

$$(x_1, \dots, x_i) : x_{i+1} \subset (x_1, \dots, x_i)^*, \qquad i = 0, \dots, d-1.$$

At least in the most important cases this theorem can be derived from 2.2 by ascending from a regular subring A to R. In fact, suppose that R is complete or the localization of an affine K-algebra. Then it can be represented as a residue class ring of a Cohen–Macaulay ring, and, furthermore, it is a finite A-module over its Noether normalization A generated by a system of parameters $x_1, \dots, x_d$; A is a regular local ring so that $x_1, \dots, x_d$ form a regular sequence in A. Thus 3.2 can essentially be regarded as a special case of the following 'colon capturing' principle:

Lemma 3.3. *Let A be a regular integral domain of characteristic p and R an integral domain that is a module-finite extension of A. Then $IR :_R JR \subset ((I :_A J)R)^*$ for all ideals I and J of A.*

The crucial point (in addition to 2.2) is that R (like every finite A-module) has a free submodule F such that $cR \subset F$ for some $c \in A$, $c \neq 0$. Then it is easy to see that c acts as the uniform annihilator required in the definition of tight closure.

Combining 3.1 and 3.2, we obtain (after a technical step) the characteristic p case of the Hochster–Roberts theorem [HR1], which has also been an important forerunner of tight closure theory:

Theorem 3.4 ([HH2]). *Let S be a regular ring of characteristic p and R a subring of S that is a direct summand. Then R is Cohen–Macaulay.*

The prime source for examples of direct summands of regular rings is invariants of group actions, and in the classical cases a linearly reductive group G acts on a polynomial ring $S = K[X_1, \dots, X_n]$ by linear substitutions. The list of linearly reductive groups in characteristic p is very short, and therefore the characteristic 0 version of 3.4 is more important for applications. Fortunately, in the setting just described the reduction to characteristic p is relatively 'easy'; see [BH, 6.5] or [Hu3, Ch. 3]. One could argue that such a reduction is superfluous because there is a better result in characteristic 0, namely, Boutot's theorem [Bou]: let K be an algebraically closed field of characteristic 0; if R is a direct summand of a finitely generated K-algebra S with rational singularities, then R has rational singularities. (We will give the definition of rational singularity below.)

Now, looking back to 3.1, we observe that it is in fact a characteristic p analogue of Boutot's theorem, suggesting that tight closure theory opens an avenue to the study of rational singularities.

4. F-rational rings and rational singularities

Let us give a name to rings in which every ideal is tightly closed:

Definition 4.1. A Noetherian ring R of characteristic p is called *weakly F-regular* if all its ideals are tightly closed. If all rings of fractions of R are weakly F-regular, then R is *F-regular*.

With this notion, Theorem 3.1 can simply be formulated as follows: direct summands of F-regular rings are F-regular. (One only has to notice that the direct summand property passes to rings of fractions.)

The distinction between weak F-regularity and F-regularity is undesirable but hard to avoid as long as the localization property of tight closure has not been proved. The most satisfactory result in this context is a theorem of Murthy: for affine algebras over uncountable fields weak F-regularity implies F-regularity (see [Hu3], Theorem 12.2).

A somewhat larger class of rings is given by the F-rational ones. Like the homological properties of being regular, Cohen–Macaulay, or Gorenstein, F-rationality is described by a property of systems of parameters:

Definition 4.2. A Noetherian local ring R of characteristic p is called *F-rational* if every ideal generated by a subset of a system of parameters is tightly closed.

For examples illuminating the distinction between F-regularity and F-rationality we refer the reader to Watanabe's papers [W1], [W2].

An F-rational ring is normal: it is an integral domain that is integrally closed in its field of fractions. Moreover, it follows immediately from 3.2 that an F-rational ring is Cohen–Macaulay, provided it is a residue class ring of a Cohen–Macaulay ring. Under this mild hypothesis, F-rationality passes to localizations: $R_{\mathfrak{p}}$ is Cohen–Macaulay along with R, and for Cohen–Macaulay rings it is necessary to test only a single system of parameters. So one chooses a system of parameters of $R_{\mathfrak{p}}$ that comes from an R-regular sequence; tight closure commutes with localization for an ideal generated by a regular sequence.

However, the notion of F-rationality (due to Fedder and Watanabe [FW]) is not just a technical concept: there is in fact a close connection between tight closure and rational singularities due to Smith [Sm1]. Let R be a local ring of an algebraic or analytic variety over an algebraically closed field of characteristic 0. One says that

R has *rational singularities* if it is normal and if for a resolution of singularities $X \to \operatorname{Spec} R$ (which is known to exist by Hironaka's theorem) the higher direct images of the structure sheaf of X vanish. If R has rational singularities, then it is Cohen–Macaulay.

The bridge between the geometric notion of rational singularity and algebraic conditions like F-rationality is provided by the notion of pseudo-rationality introduced by Lipman and Teissier (in connection with the Briançon–Skoda theorem discussed below): a local ring $(R, \mathfrak{m})$ of dimension d is said to be *pseudo-rational* if it is normal, Cohen–Macaulay, and analytically unramified, and if for every proper birational map $\pi\colon W \to \operatorname{Spec} R$ with W normal the associated map

$$H^d_{\mathfrak{m}}(\pi_* \mathcal{O}_W) = H^d_{\mathfrak{m}}(R) \to H^d_E(\mathcal{O}_W), \qquad E = \pi^{-1}(\mathfrak{m}),$$

is injective.

Theorem 4.3 ([Sm1])**.** *Let R be an excellent local ring of characteristic p. If R is F-rational, then it is pseudo-rational.*

The main step in the proof of the theorem is an analysis of the homomorphism $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ induced by the Frobenius map on R.

Suppose that R is a residue class ring of a polynomial ring $K[X_1, \dots, X_n]$ over a field of characteristic 0 with respect to an ideal generated by polynomials $f_1 \dots, f_m$. Then the $\mathbb{Z}$-algebra A generated by the coefficients of the polynomials f_i is finitely generated, and setting $R_0 = A[X_1, \dots, X_n]/(f_1 \dots, f_m)$, one has $R = R_0 \otimes_A K$. The residue class fields of A with respect to its maximal ideals are finite. One says that R has *F-rational type* if there is a Zariski open set of maximal ideals $\mathfrak{n}$ of A such that $R_0 \otimes_A A/\mathfrak{n}$ is F-rational. (This property does not depend on the choice of A.)

Theorem 4.4 ([Sm2])**.** *Let R be an affine algebra over an algebraically closed field of characteristic* 0*. If R has F-rational type, then it has rational singularities.*

The converse of Theorem 4.4 has very recently been proved by Hara [Hara]. Theorem 4.4 is not only of ideological significance. It has been used effectively by Conca and Herzog [CH] in proving that ladder determinantal rings have rational singularities.

5. The Briançon–Skoda theorem

Another elegant application of tight closure is a proof of the Briançon–Skoda theorem in characteristic p. For its statement we need the *integral closure* $\bar{I}$ of an ideal $I \subset R$: it consists of all elements $x \in R$ that satisfy an equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \qquad \text{with } a_j \in I^j;$$

$\bar{I}$ is again an ideal. It is well known that for $x \in \bar{I}$ there exists an integer k with

$$(*) \qquad x^k x^h \in I^h \qquad \text{for all } h \in \mathbb{N}.$$

Integral dependence can also be characterized by a valuative criterion: $x \in \bar{I}$ if and only if $\varphi(x) \in \varphi(I)V$ where φ is any homomorphism to a valuation domain V whose kernel is a minimal prime ideal of R. If R is Noetherian, then it is enough to consider discrete valuation domains D. Then, if $x \in I^*$, we have $\varphi(x) \in (\varphi(I)D)^* = \varphi(I)D$ where the equation holds because D is a regular local ring. We thus have proved:

Proposition 5.1. *$I^* \subset \bar{I}$.*

Of course, in general I^* is much smaller than $\bar{I}$. Nevertheless, the analogy with integral closure has proved very useful for tight closure theory.

The tight closure version of the Briançon–Skoda theorem is an asymptotic converse of 5.1:

Theorem 5.2 ([HH2])**.** *Let R be a Noetherian ring of characteristic p, and let the ideal I be an ideal generated by elements $f_1, \dots, f_n$. Then $\overline{I^{n+w}} \subset (I^{w+1})^*$ for all $w \in \mathbb{N}$.*

In the proof one must relate Frobenius powers and ordinary powers of I. This is possible through the elementary relation

$$I^{h(n+w)} \subset (f_1^h, \dots, f_n^h)^{w+1} I^{h(n-1)}.$$

We may assume that R is an integral domain. Let $x \in \overline{I^{n+w}}$. By relation $(*)$ above, there exists k with $x^k x^h \in I^{h(n+w)} \subset (f_1^h, \dots, f_n^h)^{w+1} I^{h(n-1)}$ for all $h \in \mathbb{N}$. When we choose $h = q = p^e$ and set $c = x^k$, then

$$cx^q \in (f_1^q, \dots, f_n^q)^{w+1} = (I^{w+1})^{[q]}.$$

Corollary 5.3. *Suppose in addition that R is a regular (or just a weakly F-regular) ring of characteristic p. Then $\overline{I^{n+w}} \subset I^{w+1}$, and in particular $\overline{I^n} \subset I$.*

The original Briançon–Skoda theorem [BrS] is the assertion of 5.3 for R being the convergent power series ring over $\mathbb{C}$ in d variables (and $w = 0$). In fact, one can replace n in 5.2 and 5.3 by d (or the minimum of n and d) if R is local: according to a theorem of Northcott and Rees, for every ideal I of a Noetherian local ring of dimension d (and with infinite residue class field) there exists an ideal $J \subset I$ such that $\bar{J} = \bar{I}$ and J is generated by at most d elements.

While Briançon and Skoda's proof rests on a genuinely analytic argument, algebraic proofs for 5.3 were given by Lipman and Sathaye [LS] (with R a regular local ring) and Lipman and Teissier [LT] (with R pseudo-rational and n replaced by d).

The characteristic 0 version of 5.3 can be reduced to characteristic p, but the case of mixed characteristic seems unreachable from characteristic p. Therefore the tight closure proof does not supersede the theorems of Lipman–Sathaye and Lipman–Teissier. However, it offers a refinement we have neglected so far: the extra factor $I^{h(n-1)}$ in the containment $x^k x^h \in (f_1^h, \dots, f_n^h)^{w+1} I^{h(n-1)}$. Taking it into account leads one to the *Briançon–Skoda theorems with coefficients* of Aberbach–Huneke [AH]. For another variant see Swanson [Sw].

6. Tight closure in modules

The notion of tight closure can be generalized from the situation $I \subset R$ to that of submodules $N \subset M$. Recall the Frobenius functor defined in Section 1. Let $e \in \mathbb{N}$, and write x^q for the image of $x \in M$ under the natural map $M \to \mathcal{F}^e(M)$, $m \mapsto m \otimes 1$ and $q = p^e$. Furthermore, let N_M^q denote the R-submodule generated by the x^q, $x \in N$. Then the *tight closure* N_M^* (or simply N^*) *of N in M* is the set (actually a submodule) of the elements $x \in M$ for which there exists $c \in R^\circ$ with $cx^q \in N_M^q$ for $q \gg 0$. As in the case of ideals, contractions of extensions by module finite algebras are contained in the tight closure, and so this holds also for the contraction from R^+. Furthermore, if R is regular, then all submodules of a finitely generated R-module are tightly closed.

The most important notion in this context is that of phantom acyclicity.

Definition 6.1. A complex

$$\mathbb{G}\colon \cdots \longrightarrow G_n \xrightarrow{\varphi_n} G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\varphi_0} G_0 \longrightarrow 0$$

is *phantom-acyclic* if $\operatorname{Ker}\varphi_{i-1} \subset (\operatorname{Im}\varphi_i)^*$ (in G_{i-1}) for all $i > 0$. It is *stably phantom-acyclic* if $\mathcal{F}^e(\mathbb{G})$ is phantom acyclic for all e.

The following theorem characterizes stably phantom acyclic complexes of free modules and brings this notion very close to the Buchsbaum-Eisenbud acyclicity criterion.

Theorem 6.2 ([HH2]). *Let R be a local ring that is a homomorphic image of a Cohen–Macaulay ring and such that* $\dim R/\mathfrak{p} = \dim R = d$ *for all minimal prime ideals* $\mathfrak{p}$. *Let*

$$\mathbb{F}\colon 0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_0} F_0 \longrightarrow 0$$

be a complex of finite free R-modules, and set $r_i = \sum_{j=i}^{m}(-1)^{j-i}\operatorname{rank}F_j$. Then $\mathbb{F}$ is stably phantom acyclic if and only if $d - \dim R/I_{r_i}(\varphi_i) \geq i$ for all $i = 1, \dots, m$.

Note that r_i is the rank of φ_i if $\mathbb{F}$ is acyclic (this follows easily from the additivity of rank along exact sequences); $I_{r_i}(\varphi_i)$ denotes the ideal generated by the determinants of the $r_i \times r_i$ submatrices of a matrix of φ_i. (A submatrix is the intersection of a set of columns with a set of rows.)

Let $\operatorname{grade} I$ denote the maximal length of a regular sequence contained in an ideal I; then $\operatorname{grade} I \leq d - \dim R/I$. If one even has $\operatorname{grade} I_{r_i}(\varphi_i) \geq i$ for $i = 1, \dots, m$, then the complex $\mathbb{F}$ is acyclic by the Buchsbaum–Eisenbud acyclicity criterion (for example, see [BH, 1.4.12]). Thus the replacement of $\operatorname{grade} I_{r_i}(\varphi_i)$ by the potentially larger invariant $d - \dim R/I_{r_i}(\varphi_i)$ weakens acyclicity to phantom acyclicity.

Aberbach [Ab] has developed the theory of modules of the form $M = H_0(\mathbb{F})$ where $\mathbb{F}$ is a stably phantom acyclic complex as in the theorem. Such modules are said to have *finite phantom projective dimension*, and $\mathbb{F}$ is a *phantom free resolution.* For example, the numerical invariants of a minimal phantom free resolution are determined by M, and its length, denoted by $\operatorname{ppd} M$, is given by a 'phantom' Auslander–Buchsbaum formula.

Because of the persistence of tight closure, phantom homology is mapped to phantom homology under ring extensions and in particular vanishes under maps to regular rings, a fact that explains the attribute *phantom.* From this vanishing principle one can derive the general version of the Hochster–Roberts theorem 3.4.

There are 'phantom' versions of other theorems, and in particular the *phantom intersection theorem*, improving the Evans–Griffith 'improved new intersection theorem', yields an approach to the homological conjectures that circumvents the use of big Cohen–Macaulay modules (or algebras):

Theorem 6.3 ([HH4]). *Let $(R, \mathfrak{m})$ be a complete local ring of characteristic p such that* $\dim R/\mathfrak{p} = \dim R = d$ *for all minimal prime ideals* $\mathfrak{p}$. *Let M be a module of finite phantom projective dimension m. If* $\dim R/(\operatorname{Ann} x) < d - m$ *for $x \in M$, then $x \in 0^*$. In particular* $\dim R/(\operatorname{Ann} x) \geq d - m$ *for all $x \in M \setminus \mathfrak{m}M$.*

From the improved new intersection theorem—and *a fortiori* from its phantom version—one can deduce several 'homological conjectures' in the equicharacteristic case, for example the Evans–Griffith syzygy theorem and bounds for the Bass numbers of a module (for example, see [BH, Ch. 9]).

7. Further topics

Tight closure theory has several other aspects, for example:

- F-purity (Hochster and Roberts [HR2], Fedder and Watanabe [FW]),
- tight closure in graded rings (Smith [Sm3]),
- Hilbert–Kunz functions and multiplicities (Han and Monsky [HanM]),
- uniform Artin–Rees theorems (Huneke [Hu2]),
- arithmetic Macaulayfication (Huneke and Smith [HSm1]),
- strongly F-regular rings (Hochster and Huneke [HH1], Glassbrenner [Gl]),
- differentially simple rings (Smith and Van den Bergh [SVdB]),
- regular base change ([HH5], Velez [Ve]),
- solid closure (Hochster [Ho2]),
- Kodaira vanishing (Huneke and Smith [HSm2]).

The references above have not been chosen systematically, but they should suffice as entry points to the theory. Huneke's lecture notes [Hu3] treat almost all of the topics listed and contain a comprehensive bibliography.

References

[Ab] I. Aberbach, *Finite phantom projective dimension*, Amer. J. Math. **116** (1994), 447–477. MR **95c:**13012

[AHH] I. Aberbach, M. Hochster, and C. Huneke, *Localization of tight closure and modules of finite phantom homology*, J. Reine Angew. Math. **434** (1993), 67–114. MR **94h:**13005

[AH] I. M. Aberbach and C. Huneke, *A theorem of Briançon-Skoda type for regular local rings containing a field*, Proc. Amer. Math. Soc., to appear. CMP 95:03

[Bou] J.-F. Boutot, *Singularités rationelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987), 65–68. MR **88a:**14005

[BrS] J. Briancon and H. Skoda, *Sur la clôture intégrale d'un idéal de germes de fonctions holomerphes en un point de C^n*, C. R. Acad. Sc. Paris Sér. A **278** (1974), 949–951. MR **49:**5394

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge studies in advanced mathematics **39**, Cambridge University Press 1993. MR **95h:**13020

[CH] A. Conca and J. Herzog, *Ladder determinantal ideals have rational singularities*, Adv. in Math., to appear.

[FW] R. Fedder and K.-I. Watanabe, *A characterization of F-regularity in terms of F-purity*, in Commutative Algebra, Math. Sci. Research Inst. Publ. **15**, Springer-Verlag 1989, 227–245. MR **91k:**13009

[Gl] D. Glassbrenner, *Strong F-regularity in images of regular rings*, Proc. Amer. Math. Soc. **124** (1996), 345–353.

[HanM] C. Han and P. Monsky, *Some surprising Hilbert–Kunz functions*, Math. Z. **214** (1993), 119–135. MR **94h:**13008

[Hara] N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps*, preprint.

[Ho1] M. Hochster, *Topics in the homological theory of modules over commutative rings*, C.B.M.S. Regional Conf. Ser. in Math. **24** Amer. Math. Soc. 1975. MR **51:**8096

[Ho2] ——, *Solid closure*, Contemp. Math. **159** (1994), 103–172. MR **95a:**13011

[HH1] M. Hochster and C. Huneke, *Tight closure and strong F-regularity*, Mem. Soc. Math. France **38** (1989), 119–133. MR **91i:**13025

[HH2] ——, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116. MR **91g:**13010

[HH3] ——, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. Math. **135** (1992), 53–89. MR **92m:**13023

[HH4] ——, *Phantom homology*, Mem. Amer. Math. Soc. Vol. **103**, No. **490** (1993), 1–91. MR **93j:**13020

[HH5] ——, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62. MR **95d:**13007

[HH6] ———, *Tight closures of parameter ideals and splitting in module-finite extensions*, J. Alg. Geom. **3** (1994), 599–670. MR **95k:**13002

[HH7] ———, *Applications of the existence of big Cohen-Macaulay algebras*, Adv. in Math. **113** (1995), 45–117. CMP 95:13

[HR1] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. in Math. **13** (1974), 115–175. MR **50:**311

[HR2] ———, *The purity of the Frobenius and local cohomology*, Adv. in Math. **21** (1976), 117–172. MR **54:**5230

[Hu1] C. Huneke, *An algebraist commuting in Berkeley*, Math. Intell. **11** (1989), 40–52. MR **89m:**13001

[Hu2] ———, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), 203–223. MR **93b:**13027

[Hu3] ———, *Tight closure and applications*, C.B.M.S. Regional Conf. Ser. in Math. **88**, Amer. Math. Soc. 1996.

[HSm1] C. Huneke and K. Smith, *Arithmetic Macaulayfication and tight closure*, Ill. J. Math., to appear.

[HSm2] ———, *Tight closure and the Kodaira vanishing theorem*, preprint.

[Ku] E. Kunz. *Characterizations of regular rings of characteristic p*, Amer. J. Math. **41** (1969), 772–784. MR **40:**5609

[LS] J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981), 199–222. MR **83m:**13001

[LT] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem ob Briançon-Skoda about integral closures of ideals*, Michigan Math. J. **28** (1981), 97–116. MR **82f:**14004

[PS] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. I.H.E.S. **42** (1973), 323–395. MR **51:**10330

[Ro] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. Ec. Norm. Sup. (4) **9** (1976), 103–106. MR **53:**2926

[Sm1] K. E. Smith, *F-rational rings have rational singularities*, Amer. J. Math., to appear.

[Sm2] ———, *Tight closure of parameter ideals*, Invent. Math. **115** (1994), 41–60. MR **94k:**13006

[Sm3] ———, *Tight closure in graded rings*, preprint.

[SVdB] K. E. Smith and M. Van den Bergh, *Simplicity of rings of differential operators in prime characteristic*, preprint.

[Sw] I. Swanson, *Joint reductions, tight closure, and the Briançon–Skoda theorem II*, J. Algebra **170** (1994), 567–583. MR **95m:**13003

[Ve] J. Velez, *Openness of the F-rational locus and smooth base change*, J. Algebra, to appear.

[W1] K.-I. Watanabe, *F-regular and F-pure normal graded rings*, J. Pure Appl. Algebra **71** (1991), 341–350. MR **92g:**13003

[W2] ———, *F-regular and F-pure rings vs. log-terminal and log-canonical singularities*, preprint.

Universität Osnabrück, FB Mathematik/Informatik, 49069 Osnabrück, Germany
E-mail address: `Winfried.Bruns@mathematik.uni-osnabrueck.de`