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Hochschild cohomology of von Neumann algebras, by Allan M. Sinclair and Roger R. Smith, London Math. Soc. Lecture Note Ser., vol. 203, Cambridge Univ. Press, Cambridge, 1995, vii+196 pp., \$29.95, ISBN 0-521-47880-4

A von Neumann algebra is the commutant in $B(H)$ (the algebra of bounded operators on a Hilbert space H) of a selfadjoint subset of $B(H)$, in other words, a *-subalgebra of $B(H)$ that is equal to its own bicommutant. These objects, whose investigation began with the work of Murray and von Neumann in the 30's and 40's on "rings of operators", are vastly numerous and diverse, but there is a good schematic picture of what they look like in the large, and some especially significant parts of the terrain have been mapped out in great detail. The theory of von Neumann algebras impinges on the rest of mathematics in such areas as ergodic theory, group theory, combinatorics, and mathematical physics.

Interest in the cohomology von Neumann algebras goes back about thirty years. The ingredients of the definition are a von Neumann algebra (or, for the moment, a Banach algebra) $\mathcal{M}$ and a Banach $\mathcal{M}-$bimodule $\mathcal{V}$. Write $\mathcal{L}^n(\mathcal{M},\mathcal{V})$ for the space of bounded $n-$linear maps from $\mathcal{M}^n$ to $\mathcal{V}$, and form the Hochschild complex

$$\mathcal{V} \xrightarrow{\partial} \mathcal{L}^1(\mathcal{M},\mathcal{V}) \xrightarrow{\partial} \mathcal{L}^2(\mathcal{M},\mathcal{V}) \to \dots ,$$

where the map $\partial : \mathcal{V} \to \mathcal{L}^1$ is given by $(\partial v)(x) = xv - vx$, and in higher dimensions,

$$\begin{aligned}(\partial\phi)(x_1,\dots,x_{n+1}) = \quad & x_1\phi(x_2,\dots,x_{n+1}) \\ & + \textstyle\sum_{j=1}^{n}(-1)^j\phi(x_1,\dots,x_{j-1},x_jx_{j+1},x_{j+2},\dots,x_{n+1}) \\ & + (-1)^{n+1}\phi(x_1,\dots,x_n)x_{n+1}.\end{aligned}$$

Let $H^*(\mathcal{M},\mathcal{V})$ denote the cohomology of this complex.

What one mainly wants to know about $H^n(\mathcal{M},\mathcal{V})$ in a given situation is whether or not it vanishes. For orientation, we mention a theorem of Connes [Co] characterizing injective von Neumann algebras as precisely those $\mathcal{M}$ such that $H^*(\mathcal{M},\mathcal{V})$ vanishes for all dual normal $\mathcal{M}-$bimodules $\mathcal{V}$. (To say that the bimodule $\mathcal{V}$ is dual normal means that it, like $\mathcal{M}$, is the conjugate space of a Banach space and that the module action is appropriately respectful of the w^*-topologies on $\mathcal{M}$ and $\mathcal{V}$. Hyperfinite means having a w^*-dense increasing net of finite-dimensional *-subalgebras.) The oldest result in von Neumann algebra cohomology is the Kadison - Sakai theorem [K], [S], which says that all derivations of any von Neumann algebra $\mathcal{M}$ into itself are inner; that is, if the linear map $d : \mathcal{M} \to \mathcal{M}$ satisfies $d(xy) = xd(y)+d(x)y$, then there exists v in $\mathcal{M}$ such that $d(x) = xv-vx$. This says that $H^1(\mathcal{M},\mathcal{M}) = (0)$ for all $\mathcal{M}$ (even algebraically, since automatic continuity of derivations is part of the theorem).

The central result in *Hochschild cohomology of von Neumann algebras* asserts that $H^*(\mathcal{M},\mathcal{M})$ vanishes for all $\mathcal{M}$ whose type II_1 part is stable under tensoring with the hyperfinite II_1 factor. Depending on vantage point, this either answers within epsilon the question for general von Neumann algebras posed many years ago by Kadison and Ringrose or shows the way to the last frontier. The book is addressed to readers familiar with the basics of the theory of operator algebras,

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the things one would learn in a first course on the subject. (In particular, no prior acquaintance with cohomology is assumed.) Given this prerequisite knowledge, the treatment is wholly self-contained, developing all of the techniques needed to reach the main result and to go somewhat beyond it. Rather than attempt a synopsis of the entire story told by Sinclair and Smith, we skip ahead to one of its most crucial episodes, namely the specialization of the Hochschild complex to completely bounded maps.

Complete boundedness is a notion that has found a variety of applications in the operator realm. Given a subspace $\mathcal{E}$ of a C^*-algebra $\mathcal{A}$, a bounded linear map $\phi : \mathcal{E} \to B(H)$ (or into some other C^*-algebra) is said to be completely bounded if the norms of the maps $\phi_n = \phi \otimes \mathrm{id}_n : M_n(\mathcal{E}) \to M_n(B(H))$ are bounded above, where $M_n(\mathcal{E}) = \mathcal{E} \otimes M_n$ is the space of $n \times n$ matrices with entries in $\mathcal{E}$, with norm inherited from the C^*-algebra $M_n(\mathcal{A})$. Considering all the induced maps on matrices simultaneously is a quite natural idea that gives one valuable "elbow room" in performing calculations. Christensen and Sinclair extended this concept to multilinear maps in [CS] (see also [PS]) as follows. Suppose $\mathcal{F}$ is another operator space and that $\phi : \mathcal{E} \times \mathcal{F} \to B(H)$ is bilinear. For each n, define a bilinear map

$$\phi_n : M_n(\mathcal{E}) \times M_n(\mathcal{F}) \to M_n(B(H))$$

by setting

$$\phi_n((e_{ij}), (f_{ij}))_{rs} = \sum_{k=1}^{n} \phi(e_{rk}, f_{ks}).$$

Call ϕ completely bounded if the norms of the ϕ_n's are bounded above. The definition of the maps ϕ_n mimics matrix multiplication, using $\phi(\cdot, \cdot)$ in place of operator product; with this in mind, it is clear how to define complete boundedness for multilinear maps with an arbitrary number of arguments. For suitable $\mathcal{M}-$bimodules $\mathcal{V}$, one can then form the subcomplex of the Hochschild complex that sees only completely bounded maps and thereby obtain completely bounded cohomology $H^*_{cb}(\mathcal{M}, \mathcal{V})$.

The reason this specialization is advantageous is that the cocycles can be written in a particularly simple form, thanks to the representation theorem for completely bounded maps in [CS]. Exploiting this, Sinclair and Smith present and extend results of Christensen, Effros, and Sinclair [CES] to show that $H^*_{cb}(\mathcal{M}, \mathcal{M})$ vanishes for all von Neumann algebras $\mathcal{M}$. Now the problem is to get back to norm-continuous cohomology. This proceeds by way of normal cohomology H^*_w, whose cochains respect the w^*-topologies on the various objects involved and which turns out to be the same as norm-continuous cohomology in the present setting [JKR]. The main use of the stability hypothesis in the main result is to show that certain normal multilinear maps are automatically completely bounded. Summarizing in one line the work of many people over many years, one might write

$$(0) \overset{\text{always}}{=} H^*_{cb}(\mathcal{M}, \mathcal{M}) \overset{\text{hyp.}}{\simeq} H^*_w(\mathcal{M}, \mathcal{M}) \overset{\text{always}}{\simeq} H^*(\mathcal{M}, \mathcal{M}).$$

The authors also show that $H^2(\mathcal{M}, \mathcal{M}) = H^3(\mathcal{M}, \mathcal{M}) = (0)$ when $\mathcal{M}$ is of type II_1 and has a Cartan subalgebra, and further that this much vanishing in cohomology implies that $\mathcal{M}$'s multiplicative structure is in a certain sense stable under small perturbations. The book concludes with a brief but informative appendix on bounded group cohomology.

Because of its clear and efficient presentation of material from many directions of research, *Hochschild cohomology of von Neumann algebras* is recommended reading even for operator algebraists with only a casual interest in the cohomological side of their subject.

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