BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, January 1997
S 0273-0979(97)00698-8

The heat kernel Lefschetz fixed point formula for the Spin^c dirac operator, by J. J. Duistermaat, Progr. Nonlinear Differential Equations Appl., vol. 18, Birkhäuser, Basel and Boston, MA, vii+247 pp., \$38.50, ISBN 0-8176-3865-2

Over the past few years there has been a revival of interest in the Spin^c Dirac operator from two sources: symplectic geometry and the Seiberg-Witten equations on 4-manifolds. (The main impetus behind Duistermaat's new book is the application to symplectic geometry.) On the other hand, it is the connection to K-theory which motivated the introduction of Spin^c in the late '50s and early '60s.

Recall that the special orthogonal group $SO(n)$ consists of $n \times n$ real orthogonal matrices of determinant one. It is connected but not simply connected. The connected double cover is the Lie group $Spin(n)$. For small n we can identify $Spin(n)$ with familiar groups: $Spin(2) \cong \mathbb{T}$, $Spin(3) \cong SU(2)$, $Spin(4) \cong SU(2) \times SU(2)$, $Spin(5) \cong Sp(2)$, and $Spin(6) \cong SU(4)$. Here '$\mathbb{T}$' is the circle group of unit norm complex numbers. Perhaps the most salient feature of the Spin group is its spin representation $\mathbb{S}$, which splits for even n into $\mathbb{S} \cong \mathbb{S}^+ \oplus \mathbb{S}^-$. In linear algebra a real vector space V with an orientation and inner product has a symmetry group $SO(V)$ which is isomorphic to $SO(n)$; any oriented orthonormal basis determines an isomorphism. This is the Euclidean geometry of distance, angles, and orientation. It is hard to give such a cogent description of Spin geometry. Certainly we can say that a Spin structure on V is the extra structure needed to lift the symmetry group from $SO(V)$ to a group $Spin(V)$ which is isomorphic to $Spin(n)$, and it is easy to craft a precise definition out of that idea. But the extra structure has no direct description in terms of familiar geometric notions like distance, angle, and orientation. Passing now to nonlinear smooth spaces—manifolds—we can ask for an orientation and inner product on each tangent space. Here the topology of the manifold X may introduce obstructions. There are none for the inner product, but not every manifold admits an orientation: the obstruction is the first Stiefel-Whitney class $w_1(X) \in H^1(X; \mathbb{Z}/2\mathbb{Z})$. If $w_1(X)$ vanishes, then we can make X into an oriented Riemannian manifold. Now there is an obstruction to finding a Spin structure: the second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$. If $w_2(X)$ vanishes, we can introduce a Spin structure on X.

For many applications the Spin condition is too restrictive. For example, not every complex manifold X is Spin. For such manifolds the second Stiefel-Whitney class $w_2(X) = c_1(X) \pmod 2$ is the mod 2 reduction of the first Chern class, and so $w_2(X)$ vanishes only if $c_1(X)$ is divisible by 2. Any closed oriented manifold of dimension ≤ 3 admits a Spin structure, but a closed oriented simply connected 4-manifold only admits a Spin structure if the intersection form is even. (So, for example, the complex projective plane $\mathbb{CP}^2$ is not Spin.) Spin^c is a more general structure which exists in many important geometric situations. The class of manifolds which admit Spin^c structures includes (almost) complex manifolds, symplectic manifolds, and oriented 4-manifolds. This explains the importance of Spin^c and why it has re-emerged recently in geometry and global analysis.

1991 *Mathematics Subject Classification.* Primary 58-XX.

The earliest reference I could find to the group $Spin^c(n)$ is a Bourbaki seminar exposé of Hirzebruch from 1959 [H, §5.4]. Here $Spin^c(n)$ is defined as the subgroup of $Spin(n+2)$ which lies over $SO(n) \times SO(2) \subset SO(n+2)$. Equivalently, it is the quotient of $Spin(n) \times \mathbb{T}$ by the obvious diagonal $\mathbb{Z}/2\mathbb{Z}$ subgroup. Again we can recognize $Spin^c(n)$ for small n: $Spin^c(2) \cong \mathbb{T} \times \mathbb{T}$, $Spin^c(3) \cong U(2)$, and $Spin^c(4) \cong \{\langle A, B\rangle \in U(2) \times U(2) : \det A = \det B\}$. Note that there are homomorphisms

$$Spin^c(n) \longrightarrow SO(n), \tag{1}$$

$$\det : Spin^c(n) \longrightarrow \mathbb{T}. \tag{2}$$

The first of these allows us to define the notion of a Spin^c structure on an oriented real inner product space V, just as above we defined a Spin structure. Again I do not know a simple geometric description of this structure in terms of familiar geometric notions. The second homomorphism means that a Spin^c vector space V has a canonical determinant line $\mathrm{Det}\, V$, which is a one dimensional complex hermitian vector space. Note also that the spin group $Spin(n)$ is a subgroup of $Spin^c(n)$—it is the kernel of (2)—and the spin representation $\mathbb{S}$ extends to $Spin^c(n)$. (The central $\mathbb{T}$—the kernel of (1)—acts by scalar multiplication on $\mathbb{S}$.) An oriented Riemannian manifold X admits a Spin^c structure if and only if the integral third Stiefel-Whitney class $W_3(X) \in H^3(X;\mathbb{Z})$ vanishes. This is equivalent to the existence of a class $c \in H^2(X;\mathbb{Z})$ whose reduction mod 2 is $w_2(X)$. The determinant construction, based on (2), implies that a Spin^c manifold X has a determinant line bundle $\mathrm{Det}_X \to X$ whose first Chern class c has mod 2 reduction equal to $w_2(X)$.

One final piece of background is the connection to hermitian geometry. At the most fundamental level we can say that the homomorphism $U(n) \to SO(2n)$ lifts to a homomorphism $U(n) \to Spin^c(n)$. This means that a hermitian vector space V—that is, a complex vector space with a hermitian inner product—carries a *canonical* Spin^c structure. The determinant line of the Spin^c structure is the same as the complex determinant line of V. In the nonlinear case we obtain a canonical Spin^c structure on an almost complex manifold with a hermitian metric (since each tangent space is a hermitian vector space). There are two important special cases: (i) symplectic manifolds with compatible almost complex structures and (ii) Kähler manifolds.

Returning to Hirzebruch, his celebrated Riemann-Roch theorem in the '50s led to the conclusion that certain *rational* combinations of Chern numbers on a smooth projective algebraic variety are integers. More precisely, if X is such a manifold, then the basic example is the integer

$$\mathrm{Todd}(X)[X] = e^{c/2}\hat{A}(X)[X]. \tag{3}$$

The Todd genus is initially defined in terms of Chern classes, but the right hand side expresses it in terms of the first Chern class $c = c_1(X)$ and the $\hat{A}$-genus, which is a rational combination of the Pontrjagin classes. Now any real manifold has Pontrjagin classes and the natural generalization of the first Chern class to real geometry is the determinant of a Spin^c structure. This led to the conjecture that (3) is an integer on any Spin^c manifold X.[1] The special case $c = 0$ pertains to Spin manifolds and generalizes the well-known theorem of Rohlin that the signature of

[1] At the time it was stated in terms of Stiefel-Whitney classes and the term 'c_1-manifold' was sometimes used instead of 'Spin^c manifold'.

a spin 4-manifold is divisible by 16.[2] One of the important immediate applications of the Atiyah-Hirzebruch differentiable Riemann-Roch theorem [H] is a proof of this conjecture[3] which identifies (3) as a certain integer defined in K-theory. Already in [H] the spin representation appears, but the whole story was put into a definitive context a few years later in the work of Atiyah, Bott, and Shapiro [ABS] where the group $Spin^c(n)$ is embedded in the complex Clifford algebra and the connection between Clifford algebras and K-theory is firmly established. Briefly, the Spin^c condition is the orientation condition in K-theory which one needs to define "integration". At about this time Atiyah and Singer brought elliptic operators into the game: the principal symbol of such an operator determines an element in the K-theory of the cotangent bundle. They introduced the Spin^c Dirac operator, whose principal symbol—Clifford multiplication on the spin representation—is the orientation class in K-theory. Their index theorem [AS] expresses the index of an elliptic operator as an integral in K-theory (on the cotangent bundle). Recall that the index of an elliptic operator D is defined as

$$\operatorname{index} D = \dim\ker D - \dim\ker D^*. \tag{4}$$

Finally, then, we have the more familiar interpretation of the integer (3) as the index of the Spin^c Dirac operator.

A Spin^c manifold X carries a spin bundle $S_X \to X$ determined by the spin representation. The same is true on a Spin manifold, and in that case the Levi-Civita connection determines a connection on S_X which in turn is used to define the Dirac operator. However, on a Spin^c manifold we must specify an additional piece of data: a connection on the determinant line bundle Det_X. This is important in the two applications mentioned at the beginning of the review. In symplectic geometry X is an *almost Kähler* manifold, that is, a symplectic manifold with a compatible almost complex structure. In this case there is a canonical choice for the Spin^c connection. On the other hand, in the Seiberg-Witten theory the connection on Det_X is one of the variables in the equations.

On a hermitian almost complex manifold X we identify the spinor fields with the differential forms of type $(0,q)$ (summed over q). On such forms we have what Duistermaat terms the "Dolbeault-Dirac" operator, denoted $\bar\partial + \bar\partial^*$. This is the Spin^c Dirac operator in case X is Kähler, but in general it differs. The explanation of this fact, together with an exposition of some basics about $Spin^c(n)$, Clifford algebras, and Dirac operators, occupies the first several chapters of Duistermaat's book. He emphasizes the connection with hermitian geometry, an important point of view which provides a good intuition. He carefully discusses the principal bundle of frames as it plays a crucial role in the proof of the index theorem discussed later.

The first proofs of the index theorem in the '60s relied heavily on algebraic topology—bordism or K-theory—whereas most modern accounts follow the heat equation methods pioneered by Patodi, Gilkey, and others in the early '70s (following an important paper of McKean and Singer). These heat equation proofs express the index as a difference of traces of heat kernels, which is then evaluated in the small time limit in terms of local geometry. For Dirac operators this limit exists *pointwise* and leads to a local version of the index theorem. The early accounts of this method [ABP] were aimed at the global index theorem for general operators,

[2]Actually, there is a refinement of the conjecture for spin manifolds which is the proper generalization of Rohlin's theorem.

[3]A different proof had previously been given by Milnor.

so did not emphasize this local version. Also, they used invariant theory and the computation of special cases to pin down the exact form of the index (3). Two new proofs appeared in the '80s which derive the formula (3) more directly and show more clearly where the Clifford algebra symmetry enters to give a local limit. The first is a scaling argument due to Getzler [G] which finds the $\hat{A}$-genus from Mehler's formula for the heat kernel of the harmonic oscillator. The second is an argument due to Berline and Vergne [BV] which expresses the heat kernels on X in terms of scalar heat kernels on the frame bundle. Then the $\hat{A}$-genus emerges from the formula for the differential of the exponential map on a Lie group. A few remarks: First, all of the heat equation proofs apply to any operator which can be written locally as a Dirac operator. Second, there are many important generalizations of the basic index theorem in the papers of Atiyah and Singer [AS] which for the most part have not been derived by heat kernel methods. Finally, the heat kernel method has been used to go beyond the topological invariants of these papers to define *geometric* invariants of Dirac operators (η-invariants, Quillen metrics, etc.) which have important applications in geometry.

Duistermaat's book recounts the Berline-Vergne proof of the index theorem. Following the introductory material, there are three chapters about the heat kernel. Here we find a beautiful exposition of the asymptotic expansion, though the proof that it is asymptotic to the true solution is deferred to other sources. These ideas are then applied on the principal bundle of frames, culminating in the asymptotic expansion needed to prove the index theorem. The rest of the proof relies on some linear algebra which is explained in a later chapter.

Duistermaat also treats an important generalization of the index theorem, and one which is crucial for current applications to symplectic geometry. This is the Lefschetz fixed point formula.[4] Suppose G is a compact Lie group which acts on a Spin^c manifold X by Spin^c transformations. This means that the differential of any element of G is a Spin^c transformation of the tangent spaces. Then G commutes with the Dirac operator and so acts on the kernel and cokernel. The difference of these representations is a generalization of the numerical index (4). The Lefschetz formula expresses the character of this virtual representation in terms of topological data on fixed point sets, which are smooth manifolds (whose components may have varying dimension). The Berline-Vergne proof of the index theorem extends to give the Lefschetz formula—again in terms of differential forms—as explained by Duistermaat.

Duistermaat also includes a chapter on the extension of the Lefschetz theorem to Spin^c *orbifolds*. An orbifold is a space which is locally the quotient of a smooth manifold by the action of a finite group. If M is a smooth manifold on which a compact Lie group G acts locally freely, then the quotient M/G is an orbifold. Such quotients arise in symplectic geometry as "Marsden-Weinstein reductions", and this is the main motivation behind the discussion here. I do not know of another account of the index and Lefschetz theorems for orbifolds in book form. Many readers (including this reviewer) will welcome the inclusion of this material. In the later chapters the reader will also find beautiful summaries of Chern-Weil

[4] Originally it was proved by Atiyah and Bott [AB] for isolated fixed points. Then the Atiyah-Singer index theorem [AS] was combined with the localization theorem of Atiyah-Segal [ASe] to derive the more general formula.

theory, equivariant cohomology via differential forms, and some basics of symplectic manifolds and group actions.

Duistermaat's book complements nicely the books by Berline, Getzler, and Vergne [BGV] and Roe [R]. The former is the most comprehensive and includes both modern heat equation proofs of the index theorem as well as many generalizations. Roe employs the Getzler scaling method. He discusses a basic case of the Lefschetz formula and other topics as well (such as Witten's deformation of the Morse complex and Atiyah's index theorem for infinite covers). In Duistermaat's book the student of the index theorem is fortunate to have another account of the Berline-Vergne proof (which closely follows [BGV]) together with a new set of applications. Each of these books has a different strength and a different emphasis. I found some of the notation in Duistermaat's book heavy, and there were a few small errors here and there. But the reader will be abundantly compensated by insightful observations throughout and by the later chapters on orbifolds and applications.

Let me close by briefly describing the use of Spin^c over the past few years in Seiberg-Witten theory and in symplectic geometry. (See [D] for a nice survey of early developments in Seiberg-Witten theory.) As mentioned earlier, any closed oriented 4-manifold X admits a Spin^c structure.[5] The set of all Spin^c structures is an affine space for $H^2(X;\mathbb{Z})$, and so can be identified with the set of equivalence classes of complex line bundles once a basepoint is chosen. (This is true in any dimension.) Now a Kähler surface X has a canonical Spin^c structure, and any other is obtained by specifying a complex line bundle L. In this case a solution to the (nonlinear) Seiberg-Witten equations is a holomorphic structure on L together with a holomorphic section of L,[6] or better said a divisor on X. Spin^c geometry allows a generalization of equations for a divisor—the Seiberg-Witten equations—which leads to topological invariants. The intermediate case between a general 4-manifold and a Kähler surface is a *symplectic* 4-manifold, and in an important series of papers [T1], [T2], [T3] Taubes shows that with a suitable perturbation the Seiberg-Witten equations on an almost Kähler manifold describe *pseudo-holomorphic curves*, the almost Kähler analog of divisors. This has led to many advances in symplectic topology.

The (linear) Spin^c Dirac operator has appeared recently in many papers in symplectic geometry (in arbitrary dimensions), and as mentioned above this is Duistermaat's primary reason for writing this book. Suppose X is a symplectic manifold. Let $\mathcal{C}_X$ denote the space of compatible almost complex structures; $\mathcal{C}_X$ is contractible. Now for each $J \in \mathcal{C}_X$ there is a canonical Spin^c connection and so a Spin^c Dirac operator D_J. Of course, its index is independent of $J \in \mathcal{C}_X$. It is an intriguing idea to treat the virtual vector space

$$Q(X,J) = \ker D_J - \ker D_J^* \tag{5}$$

as a "quantization" of the symplectic manifold X. For example, in case a compact group acts symplectically on X one would like to say that "quantization commutes with reduction," and as a statement about the *dimension* this was conjectured by Guillemin and Sternberg over a decade ago. (They also proved an important special case.) Recently, new techniques, notably the "symplectic cut" of Lerman [L], have been used to give a proof of this conjecture [M] using the Spin^c quantization for a

[5] This is an old result of Hirzebruch and Hopf [HH].

[6] or in some cases a holomorphic differential with values in L^*.

G-invariant choice of J.[7] But to be really useful as a quantization one needs more than the dimension of (5)—one needs the actual (virtual) Hilbert space—and so it is important to come to grips with the dependence on J. It remains to be seen if Spin^c will thus come to the fore in some new version of geometric quantization.

References

[AB] M. F. Atiyah, R. Bott, *The Lefschetz fixed point formula for elliptic complexes I*, Ann. of Math. **86** (1967), 374–407; *II*, Ann. of Math. **88** (1968), 451–491. MR **35:**3701; MR **38:**731

[ABP] M. F. Atiyah, R. Bott, V. K. Patodi, *On the heat equation and the index theorem*, Invent. math. **19** (1973), 279–330. MR **58:**31287

[ABS] M. F. Atiyah, R. Bott, A. A. Shapiro, *Clifford modules*, Topology **3** (1964), 3–38. MR **29:**5250

[AS] M. F. Atiyah, I. M. Singer, *The index of elliptic operators. I*, Ann. Math. **87** (1968), 484–530; *III*, Ann. Math. **87** (1968), 546–604; *IV*, Ann. Math. **93** (1971), 119–138; *V*, Ann. Math. **93** (1971), 139–149. MR **38:**5243; MR **38:**5245; MR **43:**5554; MR **43:**5555

[ASe] M. F. Atiyah, G. B. Segal, *The index of elliptic operators II*, Ann. of Math. **87** (1968), 531–545. MR **38:**5244

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators.*, Springer-Verlag, Berlin, 1992. MR **94e:**58130

[BV] N. Berline, M. Vergne, *A computation of the equivariant index of the Dirac operator*, Bull. Soc. Math. Fr. **113** (1985), 305–345. MR **87f:**58146

[D] S. K. Donaldson, *The Seiberg-Witten equations and 4-manifold topology.*, Bull. Amer. Math. Soc. **33** (1996), 45–70. MR **1:**339 810

[G] E. Getzler, *A short proof of the local Atiyah-Singer index theorem*, Topology **25** (1986), 111-117. MR **87h:**58207

[GGK] V. Ginzburg, V. Guillemin, Y. Karshon, *Cobordism theory and localization formulas for Hamiltonian group actions*, Internat. Math. Res. Notices **1996**, no. 5, pp. 221–234. MR **1:**383 756

[H] F. Hirzebruch, *A Riemann-Roch Theorem for differentiable manifolds*, Séminaire Bourbaki (177) (1959).

[HH] F. Hirzebruch and H. Hopf, *Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten*, Math. Annalen **136** (1958), 156–172. MR **20:**7272

[L] E. Lerman, *Symplectic cuts*, Math. Res. Lett **2** (1995), 247–258. MR **96f:**58062

[M] E. Meinrenken, *Symplectic surgery and the Spin^c-Dirac operator*, Adv. Math. (to appear).

[R] J. Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Research Notes in Mathematics Series 179, Longman Scientific & Technical, Essex, England, 1988. MR **89j:**58126

[T1] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), 809–822. MR **95j:**57039

[T2] ———, *More constraints on symplectic forms from Seiberg-Witten invariants*, Math. Res. Lett. **2** (1995), 9–13. MR **96a:**57075

[T3] ———, *SW $\Rightarrow$ Gr : from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), 845–918. MR **1:**362 874

Daniel S. Freed
University of Texas, Austin
E-mail address: dafr@math.utexas.edu

[7] Many other versions of this theorem have appeared. Incidentally, this circle of ideas has led to an improved understanding of localization in equivariant topology [GGK].