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Metrizable barrelled spaces, by J . C. Ferrando, M. López Pellicer, and L. M. Sánchez Ruiz, Pitman Res. Notes in Math. Ser., vol. 332, Longman, 1995, 238 pp., $30.00, ISBN 0-582-28703-0

The book targeted by this review is the next monograph written by mathematicians from Valencia which deals with barrelledness in (metrizable) locally convex spaces (lcs) over the real or complex numbers and applications in measure theory.

This carefully written and well-documented book complements excellent monographs by Valdivia [21] and by Pérez Carreras, Bonet [12] in this area. Although for the reader some knowledge of locally convex spaces is requisite, the book under review is self-contained with all proofs included and may be of interest to researchers and graduate students interested in locally convex spaces, normed spaces and measure theory. Most chapters contain some introductory facts (with proofs), which for the reader greatly reduces the need for original sources. Also a stimulating Notes and Remarks section ends each chapter of the book.

The classical *Baire category theorem* says that if X is either a complete metric space or a locally compact Hausdorff space, then the intersection of countably many dense, open subsets of X is a dense subset of X.

This theorem is a principal one in analysis, with applications to well-known theorems such as the Closed Graph Theorem, the Open Mapping Theorem and the Uniform Boundedness Theorem.

Spaces having the topological property in the conclusion of the Baire theorem are called Baire spaces.

Saxon [15] obtained a characterization of the Baire property for topological vector spaces: *A topological vector space X is a Baire space iff every closed balanced and absorbing subset U of X has a nonempty interior*. Now, we are very "close" to the notion of barrelledness. A lcs X is *barrelled* if every closed absolutely convex (=balanced and convex) and absorbing subset of X is a neighbourhood of zero in X.

A new line of research concerning Baire-type conditions started with Saxon in 1972. The main purpose of this research was to study the unknown stability properties of Baire locally convex spaces such as finite products and finite-codimensional subspaces of Baire spaces and to classify the normed barrelled spaces which are not Baire. Although the first approach has not been successful (finite products of normed Baire spaces need not be Baire [20] and finite-codimensional subspaces of Banach spaces need not be Baire [2]), the second one provided several new types of strong barrelledness conditions which for locally convex spaces are weaker than being a Baire space, a classification of LF-spaces and several closed graph theorems; see also Kunzinger [11].

Following Saxon [14] a lcs X is said to be *Baire-like* if given an increasing sequence of closed absolutely convex subsets of X covering X, one of the sets is a neighbourhood of zero in X.

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It was already known by an earlier result of Amemiya-Komura [1] that every metrizable and barrelled space is Baire-like; therefore within metrizable spaces the notions of barrelledness and Baire-likeness coincide.

A barrelled space is *totally barrelled* [19] if, given any covering sequence of subspaces, one of the subspaces must be barrelled and have a finite-codimensional closure.

Clearly

$$\text{Baire} \Rightarrow \text{totally barrelled} \Rightarrow \text{Baire-like} \Rightarrow \text{barrelled},$$

the converse implications do not hold. A very useful characterization of the barrelledness in metrizable spaces is the following: *A metrizable lcs X is barrelled (equivalently Baire-like) iff X is a Banach-Mackey space, i.e. every $*$-weakly bounded set is strongly bounded*. Chapter 1 collects (with proofs) some of the most important facts concerning barrelled spaces, Baire-like spaces, and totally barrelled spaces.

Much of the importance of the strong barrelledness conditions comes from their connections with fundamental principles of functional analysis. It is known (W. J. Robertson, Pták, Mahowald) that the class of barrelled spaces is the largest class of lcs for which the Open Mapping (Closed Graph) Theorem holds vis-a-vis Fréchet spaces. Also Bourbaki and Diedonne showed that the class of barrelled spaces is the largest one for which the fundamental Uniform Boundedness Theorem holds.

For Baire-like spaces one gets the following closed graph theorem (due to Saxon [14]). *If X is Baire-like, Y an LB-space with its defining sequence (Y_n) of Banach spaces and $T : X \to Y$ a linear map with closed graph, then $T(X) \subset Y_n$ for some n and T induces a continuous linear map of X into the Banach space Y_n*. Note that this theorem fails when "LB-space" is replaced by "LF-space". To compare let us recall De Wilde's closed graph theorem: *If X is the inductive limit of a family of metrizable Baire lcs and Y is a webbed space, then any linear map $T : X \to Y$ with sequentially closed graph is continuous*.

A corresponding closed graph theorem (Theorem 3.5.6 in the book) between totally barrelled spaces and spaces having an absolutely convex $\mathcal{C}$-web has been obtained by Valdivia [22].

Motivated by De Wilde's closed graph theorem, S. Dierolf and Kąkol introduced and studied [3] locally convex spaces (under the name *s-barrelled* spaces) for which every linear map into a Fréchet space with sequentially closed graph is continuous. We obtained a general construction which leads to concrete examples of linear maps into a Fréchet space with non-closed but sequentially closed graphs, and this construction applies to show that De Wilde's closed graph theorem fails when X is the inductive limit of a family of Baire bornological spaces; recall that every metrizable lcs is bornological.

The most interesting parts of the book under review are Chapters 5, 6, 8, 9, and 10, which present a survey of several interesting results, most of which have been obtained over the last several years by the authors of the book and Drewnowski in collaboration with Florencio and Paúl. Here, the question of the barrelledness (even barrelledness of class $\aleph_0$ and suprabarrelledness) of concrete non-complete metrizable (normed) spaces is discussed. Barrelled spaces of class $\aleph_0$ (originally introduced by Ferrando and López Pellicer and strictly located between Baire-like spaces and totally barrelled spaces) are used in Chapter 7 to get some applications to measure theory.

Let us recall some elementary and natural examples of non-complete normed barrelled spaces which are not Baire (Chapter 6): ℓ_0^∞ the subspace of ℓ^∞ consisting of elements $x = (x_n)$ with finite range is barrelled (of class $\aleph_0$). If Σ is a ring on a set Ω, and $\ell^\infty(\Sigma)$ is the space of Σ-simple scalar functions on Ω equipped with the sup norm, then $\ell^\infty(\Sigma)$ is barrelled iff Σ has the Nikodym-Grothendieck Property (which means that every pointwise bounded family of bounded finitely additive scalar measures on Σ is uniformly bounded). If Σ is a σ-algebra, then $\ell_0^\infty(\Sigma)$ is even barrelled of class $\aleph_0$ and suprabarrelled. Nevertheless, if Ω is infinite and Σ is the σ-algebra of all the subsets of Ω, then $\ell_0^\infty(\Sigma)$ is not totally barrelled. Chapter 6 contains several results concerning the space $\ell_0^\infty(\Sigma)$ as well as its subspaces (for instance, problems about ultrabornological property, total barrelledness, cases of nonbarrelledness).

As we have already mentioned the barrelledness of class $\aleph_0$ of the space $\ell_0^\infty(\Sigma)$ (discussed in Chapter 5) can also be used to obtain some results in measure theory. The authors obtain (Chapter 7) a generalization of Diestel and Faire's theorem to locally convex spaces (see Theorem 7.2.4) which guarantees that some finitely additive vector measures are countably additive. Also taking advantage of the barrelledness of class $\aleph_0$ of $\ell_0^\infty(\Sigma)$, one gets an extension of the Nikodym-Grothendieck fundamental boundedness principle.

Another type of result deals with the question of the barrelledness of some vector valued function spaces. In Chapter 9 the barrelledness of the space $\ell_0^\infty(\Sigma, X)$ of Σ-simple functions with values in a Hausdorff lcs X, where Σ is an infinite algebra of subsets of Ω, is considered. One gets an extension of Freniche's result: $\ell_0^\infty(\Sigma, X)$ *is barrelled iff* $\ell_0^\infty(\Sigma)$ *and* X *are barrelled and* X *is nuclear*. The following is more interesting [8]: *The space* $\ell^\infty(\Omega, X)$ *of all bounded functions on a set* Ω *with values in a normed barrelled space* X *equipped with the topology of uniform convergence is barrelled provided the cardinal number* $|\Omega|$ *or* $|X|$ *is nonmeasurable*. This interesting result covers several partial results already obtained by Ferrando (see 9.5, Notes and Remarks) and Kąkol and Roelcke [10]. Recall also that a cardinal m is said to be *measurable* if, given a set A with $m = |A|$, there exists a countably additive measure $\mu : \mathcal{P}(A) \to \{0, 1\}$ such that $\mu(A) = 1$ and $\mu(\{t\}) = 0$ for all $t \in A$. It is not known whether there exists a set with measurable cardinal. If such a set exists, then its cardinal must be strongly inaccessible.

Chapter 10 contains results (mostly obtained by the authors of the present book and Drewnowski, Florencio and Paúl) concerning barrelledness of the space $\mathcal{P}_1(\mu, X)$ of Pettis integrable functions as well as the space $\mathcal{D}_1(\mu, X)$ of Dunford integrable functions with values in a Banach space X, where (Ω, Σ, μ) is a finite measure space. It is well-known (Pettis, Thomas, Janicka and Kalton) that for μ as above and X of infinite dimension the normed space $\mathcal{P}_1(\mu, X)$ is never complete nor is it a Baire space. This chapter contains also a general (very useful) result of Drewnowski et al.; see for instance [7]:

(∗) *A quasibarrelled lcs equipped with a "suitable" Boolean algebra of projections modelled on a positive measure space without atoms of finite measure is barrelled.*

This fact provides several examples of barrelled spaces of strongly or weakly measurable functions, some spaces of vector sequences for which (∗) applies; see [7]. In particular (∗) leads to the conclusion (due to Drewnowski et al. [6], [7]) that the spaces $\mathcal{P}_1(\mu, X)$ and $\mathcal{D}_1(\mu, X)$ are barrelled. It turns out (Chapter 10, Corollary 10.6.2) that if (Ω, Σ, μ) is atomless, then both spaces are even *ultrabornological*, i.e. the inductive limits of families of Banach spaces. Note that every ultrabornological space is barrelled and for ultrabornological spaces De Wilde's closed graph theorem applies.

To complete our short discussion about Chapter 10, one should add a somewhat surprising result [6] concerning the barrelledness of the space $L_p(\mu, X)$ of Bochner p-integrable functions, where $1 \leqslant p < \infty$, X is a normed space and (Ω, Σ, μ) is finite and nonatomic: *Under the above assumptions the space $L_p(\mu, X)$ is barrelled.* The case $p = \infty$ (which needed a new idea) was solved in [5]: *If the space (Ω, Σ, μ) is additionally σ-finite, then $L_\infty(\mu, X)$ is barrelled.* It is surprising that to get the barrelledness of $L_p(\mu, X)$ we do not require X to be barrelled.

One should mention also a few words about another group of results (mainly obtained by Saxon and Sánchez Ruiz; see [16]) concerning dimensionality and cardinality of metrizable barrelled spaces and some applications. It is easy to see that *if X is a metrizable barrelled space, then either dim$(X) < \aleph_0$ or dim$(X) \geqslant \aleph_1$.* One can ask whether $\aleph_1$ is really the least possible dimension of an infinite-dimensional space of this kind. It turns out that

(i) $\mathfrak{b}$ is the *smallest* infinite-dimensionality for metrizable barrelled spaces;

(ii) $\mathfrak{b}$ is the *largest* cardinal $\mathfrak{k}$ such that every subspace of codimension less than $\mathfrak{k}$ in a metrizable barrelled space is barrelled.

Note that (i) extends an old result of Mazur about dimensionality of Fréchet spaces (which says that $\dim(X) \geqslant \mathfrak{c}$ for any infinite-dimensional Fréchet space X) and (i) combined with [17] can also be used to show that a Banach space X must have a properly separable quotient if $\aleph_0 <$ density character of $X < \mathfrak{b}$. Part (ii) extends Saxon and Levin's result [18]: *every subspace of codimension less than $\mathfrak{c}$ in a Fréchet space is barrelled, and $\mathfrak{c}$ cannot be replaced by a larger cardinal.*

In [16] (Chapter 2 in the book) the authors, replacing the cardinal $\mathfrak{c} = 2^{\aleph_0}$ with the *bounding* cardinal $\mathfrak{b}$ (already introduced in 1939 [13]), extended two BCE results of Robertson, Tweddle and Yeomans. Recall that if X is a barrelled space which does not have the finest locally convex topology and Y is a subspace of the algebraic dual X^* of X such that $Y \cap X' = \{0\}$, where X' is the topological dual of X, then the Mackey topology $\mu(X, X+Y)$ is a *countable enlargement* of the original topology $\mu(X, X')$ whenever $\dim(Y) = \aleph_0$ and is a BCE if, in addition, it is barrelled. In this case one says that X has a BCE. The unsolved BCE problem asks whether every barrelled space X with $X' \neq X^*$ has a BCE. It is known already that *no barrelled space X with dim$(X) < \mathfrak{b}$ has a BCE, but if X is a barrelledly fit space and dim$(X) \geqslant \mathfrak{b}$, then X has a BCE.* The properties of the cardinal b were used to solve the normable BCE problem without extra ZFC axiomatic assumptions: *If X is an infinite-dimensional normed barrelled space, then X has a BCE.*

This overview gives only a glimpse of the contents of this substantial book. The book treats several problems connected with barrelled spaces. Together with the previously mentioned books of Valdivia and Pérez Carreras, and Bonet, they present a complete view of the current theory. The last chapter is a short list of open problems. I read this book with great pleasure, and I warmly recommend it for everyone who is interested in this nice theory.

References

1. I. Amemiya, Y. Komura, *Über nicht vollständige Montelräume*, Math. Ann. **177** (1968), 273–277. MR **38:**508
2. J. Arias de Reyna, *Dense hyperplanes of first category*, Math. Ann. **249** (1980), 111-114. MR **81g:**46019
3. S. Dierolf, J. Kąkol, *On s-barrelled spaces*, Results in Math. **28** (1995), 40-48. MR **96f:**46002
4. J. Diestel, B. Faires, *On vector measures*, Trans. Amer. Math. Soc. **198** (1974), 253-271. MR **50:**2912
5. J. C. Diaz, M. Florencio, J. P. Paúl, *A uniform boundedness theorem for* $L^{\infty}(\mu, E)$, Arch. Math. **60** (1993), 73-78. MR **94f:**46037
6. L. Drewnowski, M. Florencio, P. J. Paúl, *The space of Pettis integrable functions is barrelled*, Proc. Amer. Math. Soc. **114** (1992), 687-694. MR **92f:**46045
7. L. Drewnowski, M. Florencio, P. J. Paúl, *Uniform boundedness of operators and barrelledness in spaces with Boolean algebras of projections*, Atti. Sem. Math. Fis. Univ. Modena **41** (1993), 317-329. MR **94m:**46004
8. L. Drewnowski, M. Florencio, P. J. Paúl, *On the barrelledness of spaces of bounded vector functions*, Arch. Math. **63** (1994), 449-458. MR **95i:**46047
9. F. J. Freniche, *Barrelledness conditions of the space of vector valued and simple functions*, Math. Ann. **267** (1984), 479-489. MR **85f:**46074
10. J. Kąkol, W. Roelcke, *On the barrelledness of* ℓ_p*-direct sums of seminormed spaces for* $1 \leqslant p \leqslant \infty$, Arch. Math. **62** (1994), 331-334. MR **95f:**46002
11. M. Kunzinger, *Barrelledness, Baire-like and (LF)-spaces*, Pitman Research Notes in Math., Longman, 1993. MR **95e:**46004
12. P. Pérez Carreras, J. Bonet, *Barrelled locally convex spaces*, Math. Studies, North-Holland, 1987.
13. F. Rothberger, *Sur un ensemble toujours de premiére catégorie qui est dépourvu de la priéte* λ, Fund. Math. **32** (1939), 294-300.
14. S. A. Saxon, *Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology*, Math. Ann. **197** (1972), 87-106. MR **46:**4140
15. S. A. Saxon, *Two characterizations of linear Baire spaces*, Proc. Amer. Math. Soc. **45** (1974), 204-208. MR **50:**10740
16. S. A. Saxon, L. M. Sánchez-Ruiz, *Barrelled countable enlargements and the bounding cardinal*, J. London Math. Soc. **53** (1996), 158-166. MR **1:**362 693
17. S. A. Saxon, A. Wilansky, *The equivalence of some Banach space problems*, Coll. Math. **37** (1977), 217-226. MR **58:**23508
18. S. A. Saxon, M. Levin, *Every countable-codimensional subspace of a barrelled space is barrelled*, Proc. Amer. Math. Soc. **29** (1971), 91-96. MR **43:**6691
19. M. Valdivia, P. Pereź Carreras, *On totally barrelled spaces*, Math. Z. **178** (1981), 263-269. MR **82m:**46002
20. M. Valdivia, *Products of Baire topological vector spaces*, Fund. Math. **125** (1985), 61-80. MR **87d:**46007
21. M. Valdivia, *Topics in locally convex spaces*, Math. Studies, North-Holland, 1982. MR **84i:**46007
22. M. Valdivia, *Classes of barrelled spaces related with the closed graph theorem*, Portugaliae Math. **40** (1981), 345-366. MR **87g:**46009

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