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Arithmetic of blowup algebras, by Wolmer V. Vasconcelos, London Math. Soc. Lecture Note Ser., vol. 195 Cambridge Univ. Press, Cambridge, 1994, 329 pp., \$34.95, ISBN 0-521-45484-0

In recent years, the theory of blowup algebras has been a highly active and successful area of commutative algebra. Wolmer Vasconcelos's book therefore comes as a timely and welcome addition to the literature. It is a fresh and lively account of classical facts as well as a rich source for recent results and original research not published elsewhere. The book serves as an excellent reference for experts, but is also accessible to graduate students with basic knowledge in commutative algebra.

The notion "blowup algebras" refers to algebraic constructions that are loosely related to the concept of blowing up a variety along a subvariety. To illustrate this concept, we describe the blowup of a plane curve at a point, which we may assume to be the origin 0. With x, y denoting coordinates of the affine plane $\mathbb{A}^2$ and s, t standing for homogeneous coordinates of the projective line $\mathbb{P}^1$ over an algebraically closed field, say $\mathbb{C}$, consider the subset Y of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by the equation $xt = ys$. Projecting Y onto $\mathbb{A}^2$, one obtains a polynomial map $\varphi\colon Y \to \mathbb{A}^2$ with $\varphi^{-1}(0) = \{0\} \times \mathbb{P}^1$, which is an isomorphism away from the "exceptional fiber" $\varphi^{-1}(0)$. The variety Y is called the blowup of $\mathbb{A}^2$ at 0. This process has the effect of replacing the point 0 by a copy of $\mathbb{P}^1$, the set of all lines in $\mathbb{A}^2$ passing through 0. Now let $C \subset \mathbb{A}^2$ be an irreducible curve containing 0. In other words, C is given by an equation $f(x, y) = 0$, for some irreducible polynomial $f(x, y) \in \mathbb{C}[x, y]$ with $f(0, 0) = 0$. One defines the *blowup* of C at 0 to be the closure Z of $\varphi^{-1}(C\backslash\{0\})$ in Y. Considering the polynomial map $\psi = \varphi_{|Z}\colon Z \to C$, one sees that Z is isomorphic to C away from the exceptional fiber $\psi^{-1}(0)$, with 0 being replaced by the distinct tangent lines to C at 0. This method of "separating tangents" plays a crucial role in the process of desingularization. In fact, a celebrated result by Hironaka asserts that over a field of characteristic zero, singularities of any dimension can be resolved by blowups. The general definition of blowups uses the notion of Rees algebras: Let R be a Noetherian ring (in the above example, $R = \mathbb{C}[x, y]/(f)$, the ring of polynomial functions defined on C), and let I be an ideal of R (in the example, $I = (x, y)/(f)$, the ideal whose common zero is the origin). To I one associates the *Rees algebra* $\mathcal{R}$, defined as the subring $R[It] \cong \bigoplus_{n=0}^{\infty} I^n$ of the polynomial ring $R[t]$, as well as the *associated graded ring* G, which is the factor ring $\mathcal{R}/I\mathcal{R} \cong \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. Now the blowup (of the prime spectrum of R along $V(I)$) is given by (the projective spectrum of) $\mathcal{R}$, with the exceptional fiber of the blowup corresponding to (the projective spectrum of) G.

The theory of blowup algebras tries to investigate various algebraic properties of $\mathcal{R}$, compare $\mathcal{R}$ and G with respect to such properties, and describe the algebra $\mathcal{R}$ in terms of generators and relations. For this purpose it is often helpful to approximate the Rees algebra by the symmetric algebra $S(I)$ of I, which maps onto $\mathcal{R}$ via a natural epimorphism $\alpha\colon S(I) \to \mathcal{R}$. The symmetric algebra of any finitely generated R-module E can be characterized in the following way: If $\{e_1, \ldots, e_n\}$

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is a generating set of E, then $S(E) = R[T_1, \dots, T_n]/L$, where $R[T_1, \dots, T_n]$ is a polynomial ring and L is the ideal generated by all linear polynomials $\sum_{i=1}^n a_i T_i$ with $\sum_{i=1}^n a_i e_i = 0$. Thus one obtains an explicit description of $\mathcal{R}$ in terms of generators and relations provided that α is an isomorphism, in which case I is said to be of *linear type* (because the relations defining $\mathcal{R}$ are linear polynomials). This leads to the problem of finding conditions ensuring the linear type property, other than the simple observation that an ideal I in an integral domain is of linear type if and only if $S(I)$ is a domain. The first one to address this question was Micali, who showed in the sixties that any ideal generated by a regular sequence is of linear type [9]; here one says that a sequence $a_1, \dots, a_n$ in R is *regular* if a_i is a nonzerodivisor on $r/(a_1, \dots, a_{i-1})$ for every i with $1 \leq i \leq n$. In the late seventies Huneke introduced the notion of a d-sequence, which greatly generalizes the concept of a regular sequence, and proved that d-sequences still generate ideals of linear type [6, 7]. Slightly later, Herzog, Simis, and Vasconcelos developed a homological device that would turn out to be a powerful tool for studying blowup algebras. They introduced the "Approximation Complexes" $\mathcal{Z}.$ and $\mathcal{M}.$, which are finite complexes of $R[T_1, \dots, T_n]$-modules whose zeroth homology are $S(I)$ and $S(I/I^2)$, respectively [2]. (Under some mild assumptions) they proved that the complex $\mathcal{M}.$ is acyclic if and only if I is generated by a d-sequence, which leads to concrete conditions for an ideal to be of linear type. The Approximation Complexes fail to be complexes of free modules, but for large classes of ideals they are sufficiently well behaved to yield information about the Cohen-Macaulay and Gorenstein properties of $\mathcal{R}$ and G, at least in the presence of the linear type condition.

Besides giving careful consideration to ideals of linear type, the author also addresses the general case where the epimorphism $\alpha\colon S(I) \to \mathcal{R}$ may have a nontrivial kernel, which needs to be determined if one is to find an explicit presentation of $\mathcal{R}$. This can be done for special classes of ideals using ad hoc methods largely based on the notion of a "Jacobian dual". When computing the kernel of α, it is also helpful to know beforehand how many homogeneous polynomials are needed to generate this ideal and what their degrees might be, information that is more readily available if the rings $\mathcal{R}$ or G are Cohen-Macaulay.

Having mentioned Cohen-Macaulayness twice, we wish to explain this notion, which plays a central role in commutative algebra. A Noetherian ring S is called *Cohen-Macaulay* if each of its localizations is Cohen-Macaulay, thus reducing us to the case where S is local, i.e., has only one maximal ideal m. For such a ring, the depth, $\operatorname{depth} S$, is defined to be the maximal length of a regular sequence contained in m, whereas the Krull dimension, $\dim S$, is the maximal length of a chain of prime ideals in S, diminished by 1. One always has the inequality $\operatorname{depth} S \leq \dim S$, and S is called Cohen-Macaulay if $\operatorname{depth} S = \dim S$. To illustrate the far-reaching consequences of this notion, let S be a Noetherian local ring of dimension d containing a field. After passing to the so-called completion of S and invoking Cohen's Structure Theorem, we may assume that S has a subring $R = k[[x_1, \dots, x_d]]$ which is a power series ring over a field and that S is a finitely generated R-module. Now S is a Cohen-Macaulay ring if and only if S is free as an R-module. In other words, Cohen-Macaulayness means that the ring has a trivial module structure over a ring as simple as a power series ring.

The Cohen-Macaulay property of blowup rings had not been studied much outside the realm of ideals of linear type until Huckaba and Huneke turned to this problem in the early nineties [4, 5]. To describe their work, let R be a Noetherian local ring of Krull dimension d having an infinite residue field. If I requires more than d generators, then I cannot be of linear type. To remedy the situation, Huckaba and Huneke pass to a minimal reduction I, thereby lowering the number of generators of the ideal. The notion of a reduction had been introduced by Northcott and Rees in the fifties [11]: An ideal J contained in I is called a *reduction* of I if the inclusion of Rees algebras $\mathcal{R}(J) \subset \mathcal{R}(I)$ makes $\mathcal{R}(I)$ a finitely generated module over $\mathcal{R}(J)$, or equivalently, if $I^{r+1} = JI^r$ for some $r \geq 0$; the smallest such r is denoted by $r_J(I)$. A *minimal reduction* of I is a reduction that is minimal with respect to inclusion, and the *reduction number* $r(I)$ of I is the minimum of the numbers $r_J(I)$ where J ranges over all minimal reductions of I. Every minimal reduction J of I can indeed be generated by at most d elements, and under suitable assumptions $\mathcal{R}(J)$ can be shown to be Cohen-Macaulay. One should think of J as a "simplification" of I, with $r(I)$ measuring how "closely" the two ideals are related. If $r(I)$ is "small enough" one might hope that the Cohen-Macaulay property passes from $\mathcal{R}(J)$ to $\mathcal{R}(I)$. Indeed, Huckaba and Huneke were able to treat the case $r(I) \leq 2$ (under suitable additional assumptions), and Wolmer Vasconcelos presents his own elegant proofs of their results, making clever use of the Approximation Complexes. After the book was completed, several authors obtained far-reaching generalizations, giving comparatively short proofs of general sufficient conditions for $\mathcal{R}$ and G to be Cohen-Macaulay. That these facts had to remain unreported simply attests to the vigor and success of the subject. The more recent results about Cohen-Macaulayness are largely based on linkage theory or, more generally, on the theory of residual intersections (as a first approximation, one can say that an ideal K is a residual intersection of I if $I \cap K$ requires as few generators as permitted by $\dim R/K$). There is yet another reason why the book devotes a whole chapter to linkage theory: Linkage provides a natural context in which the Approximation Complexes are well-behaved or, more generally, in which many results about blowup algebras apply.

We now turn to the relationship between the Rees algebra $\mathcal{R}$ and the associated graded ring G, exemplified by their respective behavior vis-à-vis Cohen-Macaulayness. It is easy to see that if R and $\mathcal{R}$ are Cohen-Macaulay, then so is G (at least if I contains a nonzerodivisor) and that the Cohen-Macaulayness of G implies the Cohen-Macaulayness of R (at least if R is local and $I \neq R$). On the other hand, $\mathcal{R}$ may fail to be Cohen-Macaulay even if G has this property (as can be seen from the example at the beginning, assuming that $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$). After the completion of the book, Lipman was able to prove that this phenomenon cannot occur in case R is a regular local ring (or more generally, a pseudo-rational local ring), such as a power series ring or a localization of a polynomial ring over a field (or the ring R from the above example, localized at the origin, as long as $\frac{\partial f}{\partial x}(0,0) \neq 0$ or $\frac{\partial f}{\partial y}(0,0) \neq 0$) [8]. His result can be paraphrased by saying that if under the above assumptions on R, the associated graded ring G is Cohen-Macaulay, then necessarily $r(I) < \dim R$ (unless R is a field). It remains to be seen what further restrictions the Cohen-Macaulayness of G imposes on the reduction number. This would provide necessary conditions for the blowup algebras to be Cohen-Macaulay, which could then be combined with the sufficient conditions described above.

Besides Cohen-Macaulayness, the book investigates other algebraic properties of $\mathcal{R}$ and G, such as Serre's conditions (S_k), Gorensteinness, integrality, and normality. A similar program is carried out for the symmetric algebra $S(E)$ of a finitely generated R-module E. How difficult and intriguing the relationship is between E and $S(E)$ can be seen from the "Factorial Conjecture", which asserts that the factoriality of $S(E)$ imposes severe conditions on the module E: If R is a regular local ring and $S(E)$ is a unique factorization domain, then does E have projective dimension at most 1; i.e., is there an exact sequence $0 \to F_1 \to F_0 \to E \to 0$ with F_0, F_1 free R-modules [3]? (Conversely, if the projective dimension of E is at most 1, one has a complete characterization of when $S(E)$ is a unique factorization domain.) This intriguing question has been answered in many cases but remains open in general. On the other hand, passing to the graded bidual $B(E)$ of $S(E)$, one always obtains a unique factorization domain, but this ring may fail to be Noetherian, as was shown by Roberts [13].

A construction similar to the one that yields $B(E)$, but applied to the Rees algebra of a (prime) ideal I, yields the *symbolic Rees algebra* $\mathcal{R}_s$ of I. This algebra is defined as $\mathcal{R}_s = \bigoplus_{n=0}^{\infty} I^{(n)} t^n \subset R[t]$, where $I^{(n)} = \{x \in R | ax \in I^n$ for some $a \in R \backslash I\}$ denotes the n-th symbolic power of the prime ideal I (which is the I-primary component of the ordinary n-th power). One always has $\mathcal{R} \subset \mathcal{R}_s$, but again, the larger algebra may fail to be a Noetherian ring, or equivalently, a finitely generated R-algebra. It is this possible lack of Noetherianness that relates the symbolic Rees algebra to two important questions: Hilbert's Fourteenth Problem and the set-theoretic generation problem. Hilbert's Fourteenth Problem can be paraphrased as follows: If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field and if K is a subfield of the field of rational functions $k(x_1, \ldots, x_n)$ which contains k, then is the intersection $R \cap K$ a finitely generated k-algebra? In the fifties, Nagata produced a counterexample based on the fact that the symbolic Rees algebra of a certain ideal I is not Noetherian (his ideal I is not prime though; it is the set of polynomials vanishing on 16 general lines in $\mathbb{A}^3$ passing through 0) ([10]; see also [12]). In its simplest form, the set-theoretic generation problem addresses the question of when a variety $V \subset \mathbb{A}^n$ of codimension $c = n - \dim V$ is, locally as a set, the intersection of c hypersurfaces, the codimension being the intuitively expected number of hypersurfaces needed. The corresponding algebraic question can be formulated as follows: Let R be a regular local ring with infinite residue field, and let I be a prime ideal of codimension $c = \dim R - \dim R/I$; when is it possible to find c elements $f_1, \ldots, f_c$ in I so that $I = \sqrt{(f_1, \ldots, f_c)}$, where $\sqrt{J} = \{x \in R | x^n \in J$ for some $n > 0\}$ denotes the radical of an ideal J? Cowsik and Vasconcelos observed that this is indeed possible for $\dim R/I = 1$ (i.e., for curves) as long as $\mathcal{R}_s$ is Noetherian, triggering the question of whether the symbolic Rees algebra of prime ideals in regular local rings might always be Noetherian. This problem has a negative answer in general even if R is a power series ring over a field, as was shown by Roberts and by Goto, Nishida, and Watanabe in the early nineties [13, 1]. The latter authors even produce counterexamples with $\dim R/I = 1$. To describe one of their examples, consider the subring $S = k[[t^{25}, t^{29}, t^{72}]]$ of the power series ring $k[[t]]$ over a field, and let I be the kernel of the natural map from the power series ring $R = k[[x, y, z]]$ onto S; then the symbolic Rees algebra $\mathcal{R}_s$ of I is not Noetherian if char $k = 0$. On the other hand, $\mathcal{R}_s$ is Noetherian whenever char $k > 0$, a fact that is actually used in the proof of non-Noetherianness in characteristic 0 [1]! These fascinating developments are reported in the book under review, as well as various positive results concerning the Noetherian property of $\mathcal{R}_s$.

This review gives a small sample of the numerous topics and viewpoints presented in the text. Interested readers should have a closer look at this excellent book.

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