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Fully nonlinear elliptic equations, by Luis A. Caffarelli and Xavier Cabré, Amer. Math. Soc. Colloq. Publ., vol. 43, Amer. Math. Soc., Providence, RI, 1995, vi + 104 pp., \$39.00, ISBN 0-8218-0437-5

One of the major advances in the theory of partial differential equations during the last twenty years has been the development of techniques for studying fully nonlinear second-order elliptic equations. These are equations of the general form

$$F[u] = F(D^2u, Du, u, x) = 0 \tag{1}$$

where F is nonlinear with respect to D^2u as well as possibly with respect to u and Du. Ellipticity means that F is a monotone function of the second derivative variables, in the sense that for any $(M, p, z, x) \in \mathbf{S}^{n\times n} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, where $\mathbf{S}^{n\times n}$ denotes the space of $n \times n$ real symmetric matrices, we have

$$F(M + N, p, z, x) > F(M, p, z, x) \tag{2}$$

for any positive definite $N \in \mathbf{S}^{n\times n}$. In addition, F is said to be uniformly elliptic if for any positive $N \in \mathbf{S}^{n\times n}$ we have

$$\lambda\|N\| \le F(M + N, p, z, x) - F(M, p, z, x) \le \Lambda\|N\| \tag{3}$$

for all $(M, p, z, x) \in \mathbf{S}^{n\times n} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, where λ and Λ are some positive constants, called the ellipticity constants of F. Examples of such equations are the Bellman equation

$$F[u] = \inf_{\alpha\in\mathcal{A}} \{L_\alpha u - f_\alpha(x)\} = 0 \tag{4}$$

and Isaacs' equation

$$F[u] = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \{L_{\alpha\beta} u - f_{\alpha\beta}(x)\} = 0, \tag{5}$$

where each $L_\alpha, L_{\alpha\beta}$ is a linear elliptic operator of the form

$$L = a_{ij}(x)D_{ij} + b_i(x)D_i + c(x). \tag{6}$$

These are uniformly elliptic if each $L_\alpha, L_{\alpha\beta}$ is uniformly elliptic with ellipticity constants independent of α and β. Bellman and Isaacs' equations arose originally in stochastic control theory and stochastic games theory, but it turns out that many equations arising elsewhere can also be written in these forms.

Two main strategies have been developed for solving fully nonlinear elliptic equations. One approach is to prove the existence of classical solutions of, say, the Dirichlet problem in a smooth bounded domain $\Omega \subset \mathbf{R}^n$ directly using the continuity method. For this, one needs to prove a priori estimates for solutions in the space $C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$; i.e., one needs to bound u and its derivatives up to second order as well as the α Hölder seminorm of the second derivatives. This approach has led to the existence of classical solutions for a wide variety of fully nonlinear elliptic equations subject to various boundary conditions, but without doubt the central results are the interior second derivative Hölder estimates of Evans [6, 7] and Krylov [12] and the corresponding global estimate of Krylov [13].

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These results in turn depend on the Harnack inequality for linear elliptic equations in nondivergence form due to Krylov and Safonov [15, 16].

The second approach is to prove the existence of some kind of generalized solutions and then to establish their uniqueness and regularity. The concept of generalized solution which has evolved in the work of Evans [4, 5], Crandall and Lions [3], and Lions [17] is that of viscosity solution, although for specific classes of equations there are also other notions. A continuous function u defined on a domain Ω in $\mathbf{R}^n$ is said to be a *viscosity subsolution* (respectively, *viscosity supersolution*) of the elliptic equation

$$F(D^2u, x) = 0 \tag{7}$$

if for any C^2 function ϕ on Ω and any local maximum (respectively, minimum) $x_0 \in \Omega$ of $u-\phi$ we have $F(D^2\phi(x_0), x_0) \geq 0$ (respectively, ≤ 0). A *viscosity solution* is a continuous function that is both a viscosity subsolution and supersolution. The point is that the test function ϕ should satisfy the inequalities which would hold by virtue of the maximum principle and the ellipticity of the equation if u were a C^2 solution. The notion of viscosity solution turns out to be very useful, because it is stable under locally uniform convergence of both u and F and because existence and uniqueness results for such solutions can be proved under very general conditions, and in particular for equations for which the existence of classical solutions is not known (and perhaps not true), such as Isaacs' equation in dimensions greater than two.

A major breakthrough in the theory of viscosity solutions was made by Jensen [11], who proved a comparison principle which implied the uniqueness of viscosity solutions of the Dirichlet problem for (7), at least for F independent of x. Later refinements allowed this assumption to be relaxed. Using these results, Ishii [9, 10] observed that the existence of viscosity solutions of the Dirichlet problem followed from the Perron method.

Finally we come to the main topic of this book, which is the regularity theory for viscosity solutions of uniformly elliptic equations of the general form

$$F(D^2u, x) = f(x). \tag{8}$$

It is based on a series of lectures given at New York University in 1993. The central results are the $W^{2,p}, C^{2,\alpha}$, and $C^{1,\alpha}$ estimates of the first author [1, 2]. To describe these, it is probably best to follow the book and to recall the corresponding estimates for linear equations. Let u be a bounded solution of the uniformly elliptic equation

$$Lu = a_{ij}(x)D_{ij}u = f(x) \tag{9}$$

in the unit ball $B_1 \subset \mathbf{R}^n$. Then the following are true:

(i) (Cordes-Nirenberg type estimates) Let $0 < \alpha < 1$ and suppose that

$$\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \delta(\alpha)$$

for sufficiently small $\delta(\alpha) > 0$. Then $u \in C^{1,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}).$$

(ii) (Schauder estimates) If a_{ij} and f belong to $C^\alpha(\overline{B}_1)$, then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)}).$$

(iii) (Calderón-Zygmund estimates) If a_{ij} are continuous in B_1 and $f \in L^p(B_1)$ for some $1 < p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}).$$

These estimates are obtained by first deriving the estimate for solutions of $\Delta u = f$ and then using a perturbation technique. The idea in the nonlinear case is similar: if one has suitable existence results and interior estimates for solutions of the "constant coefficient" equation

$$F(D^2u, x_0) = f(x_0), \tag{10}$$

then under certain assumptions on F and f one can derive $C^{1,\alpha}, C^{2,\alpha}$, or $W^{2,p}$ estimates for viscosity solutions of (8) by a perturbation argument. The basic assumption on F required to carry out this procedure is that the quantity

$$\beta(x, x_0) = \sup_{M \in S^{n\times n}} \frac{|F(M,x) - F(M,x_0)|}{\|M\|}$$

is sufficiently small if $|x - x_0|$ is small. The precise nature of this smallness condition and the estimates required for solutions of (9) depend on which estimate one is considering, but in each case the smallness of $\beta(x, x_0)$ is measured in the L^n norm rather than the L^∞ norm. The techniques thus give improved versions of the classical linear estimates stated above.

The book begins with some preliminary material concerning tangent paraboloids and second-order differentiability. In Chapter 2 viscosity solutions are introduced, and the class $\mathcal{S}(\Lambda, \lambda, f)$ of "all viscosity solutions of all elliptic equations of the form (8) with ellipticity constants Λ and λ" is defined using Pucci's extremal operators. This is important in what follows because it allows the authors to avoid the traditional Bernstein method of differentiating the equation to obtain linear differential inequalities for the derivatives of the solution. Clearly, this procedure is not possible if F and u are not sufficiently smooth.

Chapters 3 and 4 deal with two crucial tools from the linear theory. These are the Alexandroff-Bakelman-Pucci estimate and maximum principle and the Harnack inequality of Krylov and Safonov. These are proved for viscosity solutions rather than classical solutions, so the proofs are a little more complicated in certain parts than those presented elsewhere, for example in [8] or [14]. An important consequence of the Harnack inequality is the C^α interior regularity of solutions of (8).

The following two chapters deal with the existence and uniqueness results and estimates for solutions of uniformly elliptic equations of the form

$$F(D^2u) = 0 \tag{11}$$

which are necessary to carry out the perturbation procedure mentioned above. A proof of Jensen's comparison principle for viscosity solutions of (11) is given, but the existence of solutions of the Dirichlet problem is not proved. It is only remarked that the Perron method can be used for this once one has a comparison principle. A $C^{1,\alpha}$ interior estimate for solutions of (11) is proved, and in Chapter 6 a version of the Evans-Krylov $C^{2,\alpha}$ interior estimate for C^2 (in fact, even $C^{1,1}$) solutions of concave equations of the form (11) is presented. In addition, a new proof of the $C^{1,1}$ interior regularity of viscosity solutions of such equations is given.

In the following two chapters these estimates are combined with delicate perturbation arguments to obtain the $W^{2,p}, C^{2,\alpha}$, and $C^{1,\alpha}$ estimates mentioned above. This is the most technical part of the book and cannot be described in detail here. The key idea, however, is to consider paraboloids of the form $P(x) = u(x_0) + l(x - x_0) \pm \frac{1}{2}M|x|^2$, where l is a linear function, and to show that the measure of the complement in $B_{1/2}$ of the set of points x_0 at which there is such a paraboloid touching the graph of u from above (or from below) must decay sufficiently quickly as M gets large. This gives control of the distribution function of second-order difference quotients of u, leading to $W^{2,p}$ estimates. The $C^{2,\alpha}$ (respectively, $C^{1,\alpha}$) estimates are proved by showing that the existence and regularity results for solutions of (10) imply that the solutions of (8) are well approximated by quadratic polynomials (respectively, affine functions).

In the final chapter the authors present an alternative proof of the $C^{1,1}$ interior estimate for smooth solutions of concave equations of the form (11). In addition, they describe the proof of the classical solvability of the Dirichlet problem for such equations using the continuity method. Most of the necessary estimates are proved in detail, but the proof of Krylov's boundary gradient Hölder estimate is omitted.

The book is well written, with the arguments clearly presented. There are helpful remarks throughout the book, and at several points the authors give the main ideas of the more technical proofs before proceeding to the details. No previous knowledge of viscosity solutions is assumed, but readers who are not familiar with the existence and uniqueness theory of viscosity solutions will probably want to consult other sources for this, as these aspects are not covered in detail. The book will certainly be of interest to researchers and graduate students in the field of nonlinear elliptic equations.

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