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Selberg zeta and theta functions, A differential operator approach, by U. Bunke and M. Olbrich, Mathematical Research, vol. 83, Akademie Verlag GmbH, Berlin, 1995, 168 pp., DM 78, ISBN 3-05-501690-4

Let X be a Riemannian manifold on which a discontinuous group of isometries Γ acts; a typical example would be the universal cover of a Riemannian manifold Y and Γ subgroup of the fundamental group of Y. A. Selberg discovered in the early 1950s that if X is a Riemannian symmetric space—that is, if the connected component of the isometry group G is a Lie group acting transitively on X—then there is a very strong connection between the nature of Γ as a subgroup of G and the eigenvalues of the ring of invariant differential operators on X for eigenfunctions invariant under Γ. One expression of this relationship is the Selberg Trace Formula. This is a distributional identity between, on the one side, a sum parametrized by conjugacy classes of Γ and, on the other, a sum parametrized by certain eigenvalues. Another way of looking at this, first introduced by Gelfand and Piatetski-Shapiro, is to regard it as a character formula, namely, the equality between two expressions for the representation of G obtained by inducing the identity representation of Γ to G.

Selberg noticed that the structure of the Trace Formula in certain cases (for example, when $G \cong \mathrm{SL}_2(\mathbb{R})$) is similar to that of the explicit formulae of Prime Number Theory. He was led to introduce a certain analytic function, now called the Selberg Zeta Function, which is analogous to the Riemann Zeta Function and its generalizations. This function is defined as an infinite product over the conjugacy classes of Γ, convergent in a half-plane. Selberg showed, in certain cases, that it had an analytic continuation to the complex plane as a meromorphic function. Its zeros and poles are spectral invariants.

There is an extensive literature devoted to the theory of the Selberg Zeta Function, and it might seem that this is a chapter of mathematics which should have been closed years ago with no further need for a research monograph like the book under review. As is so often the case, the situation is not as simple as it may seem. First of all, in determining the properties of the Selberg Zeta Function when Γ is cocompact and X is of rank 1 (i.e., there are no 2-dimensional flat submanifolds in X), one actually proves first that the logarithmic derivative is meromorphic; this follows from an application of the Trace Formula. In order to prove that the Selberg Zeta Function itself is meromorphic, one has to show that the residues at the poles are integral, and this is not easy. Selberg told the reviewer in 1987 that he had studied many cases and that this appeared always to be the case. Indeed, it was a stumbling block for many years, and it was first completely resolved along these lines by A. Juhl [1]. The underlying problem is one of integral geometry; the values of certain integrals have to be shown to be integral up to known factors. The best-known example of this phenomenon is the Gauss-Bonnet Theorem, but many more appear, for example, in and around the Atiyah-Singer Index Theorem.

A second development which forced a fundamental reconsideration of the theory of the Selberg Zeta Function came from quite a different quarter, namely, from the

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physics of dynamical systems. David Ruelle proposed in 1976 a new method for studying such zeta functions and gave an entirely different proof of the meromorphic continuation of the Selberg Zeta Function. He did not treat this theory in detail; this was done by David Fried in 1986. In particular the latter introduced a new definition of the Selberg Zeta Function. From Selberg's method it had been clear that one could consider the Selberg Zeta Function "twisted" (a) by a representation of the maximal compact subgroup K of G and (b) by a unitary representation of Γ itself. Fried discovered that it is much more appropriate to consider in case (a) representation of a subgroup M of K, namely, that which preserves a fixed direction. In other words, he shifted the emphasis from the symmetric space to the sphere-bundle over this space, which reflects the fact that the geodesic flow (the dynamical system here) operates on this space. At this point it really became clear what the goals should be. Moreover, Ruelle and Fried's method gave the meromorphic continuation of the Selberg Zeta Function without any appeal to integral geometry. A new set of questions arose in trying to understand how Selberg's and Ruelle's methods relate to one another. It is worth noting at this stage that twists of type (b) can be encompassed by looking at the appropriate embedding of Γ in $G \times U(n)$, which is also of real rank 1 (but is not simple); thus it is worthwhile to study arbitrary groups of rank 1, ones which include those of the type $G \times U(n)$.

The point of this monograph is to give a complete account of a very general class of Selberg Zeta Functions using methods from differential geometry rather than the more group-theoretic ones of Juhl. The generality consists in the fact that the space X and the representation of M are as general as possible. On the other hand, the authors restrict their attention to the case where the group Γ is torsion-free and cocompact (i.e., $\Gamma\backslash X$ is compact). The restriction to torsion-free groups is to a large extent a product of the philosophy which the authors have espoused and means that $\Gamma\backslash X$ should be a manifold. They have given a very beautiful description as to how Selberg's theory fits together with more modern methods of global analysis, especially those which were developed in connection with the Atiyah-Singer Index Theorem. In particular they describe in detail how one can introduce operators of Laplace-Beltrami type which depend on a representation of the group M. This is of crucial importance in obtaining the generality that one knows has to exist. Having done this, they solve the problems of integral geometry by following the philosophy of the Hirzebruch Proportionality Principle. The main technical tool used in this is the fundamental solution of the wave equation which can be used to give a close relationship between X and a "compact dual" which depends only on G. The Selberg Zeta Function can be treated in this case as well—this was done by V. Schubert in his Göttingen dissertation using the method of Gelfand and Piatetski-Shapiro, as developed further by Langlands and Osborne, and which was also that used by Juhl. The restriction to groups without parabolic elements has been partly occasioned by the same considerations but also by the inherent technical complexity. The authors discuss briefly what happens when there are parabolic elements when X is the hyperbolic plane. They do not describe this in detail, and a good description of the divisor in this case would be very welcome.

In fact, it seems as if a large part of this theory can be extended much further. The assumption that $\Gamma\backslash X$ need be compact, or at least of finite volume, can be replaced by a much weaker condition which allows for very non-compact manifolds. At the present time a fairly complete understanding is being built up of the case where Γ is "convex cocompact" (which means roughly that there is a fundamental domain with finitely many sides but that there are no parabolic elements in Γ); the two authors of this monograph have made considerable contributions here, which are briefly alluded to in the final chapter. The most general case which one expects to be able to understand properly is when Γ is "geometrically finite" (the same condition as above but now allowing parabolic elements), but there are still very considerable technical obstacles in the way of progress here.

The theta functions in the title of the book are those associated with the names of Minakshisundaram and Pleijel which form generating functions for the eigenvalues of certain elliptic operators. There are many other classes of functions which have been given this name, and they bear little relation to one another. These play an important role in the differential geometry surrounding the Atiyah-Singer Index Theorem and the Atiyah-Bott-Lefschetz Fixed Point Theorem. They also allow another way of describing the information contained in the Selberg Trace Formula. This is essentially the goal here. The authors use this function to show what happens in the presence of parabolic elements, but only in the special case described above.

It hardly needs to be said that this book is not designed as a beginner's guide to the Trace Formula—for most people it would be much better to start with the case of subgroups of $\mathrm{PSL}_2(\mathbb{R})$ (of which there are many accounts)—but the proofs in this book are self-contained, and to work through them would be a very instructive way of learning not only about the Selberg Trace Formula but also about techniques of global analysis.

The reader may be asking whether this sort of generality has any real point: does anyone really need to know about the analytic properties of the Selberg Zeta Function? With the usual reservations about the importance of mathematics for the world at large, there are two serious reasons why these results are of importance. The first lies in the comparison between the mathematics of classical physics and of quantum theory. The definition of the zeta function is "classical" in that it deals with the lengths of closed orbits of the geodesic flow. The poles and zeros of the Selberg Zeta Function depend on spectral data and are therefore "quantum mechanical" in nature. This relationship is fascinating, but it is special to the case in hand: one would normally expect an asymptotic relationship where "Planck's constant" tends to zero, i.e., a relationship between the distribution of the geodesics and the large-scale asymptotic behaviour of the eigenfunctions of the corresponding Laplace-Beltrami operator. Nevertheless, the mathematical background of the Correspondence Principle is far from being fully understood, and these examples are therefore very valuable.

The other reason for the interest in this theory is that through Ruelle's work (and that of Fried and Juhl) it has become clear that the theory of the Selberg Zeta Function represents an analogue of the Lefschetz Fixed Point Formula for flows. Such flows have to be rather special, at least at the moment, in that they will have to satisfy Smale's Axiom A. Nevertheless, this is a development that appears as if it could have wider applications. The theory of the Ruelle Zeta Function (a close relative of Selberg's but in a much more general context) has been developing rapidly recently. Although it is clear that the central Manning's Lemma is cohomological in nature, it has proved very difficult to bring this out in a formal theory. However, it does seem as if A. Connes's ideas of an Index Theory for foliations are pertinent here. Ruelle's theory suggests what one might expect in a more general context when one attempts to understand more general flows. In the last chapter the authors discuss very briefly some ideas in this direction and some of their recent work that goes a long way towards making this idea concrete.

This book provides a very readable introduction to the borderline between differential geometry and harmonic analysis as well as a much-needed reference for the properties of the Selberg Zeta Function for discrete subgroups of Lie groups of real rank 1.

References

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