

Cogroups and co-rings in categories of associative rings, by George M. Bergman and Adam O. Hausknecht, Mathematical Surveys and Monographs, vol. 45, American Mathematical Society, Providence, RI, 1996, x + 388 pp., \$79.00, ISBN 0-8218-0495-2

Any primer of category theory will tell you that a representable functor is a functor F on an arbitrary category \mathcal{C} to \mathbf{Set} which is isomorphic (naturally equivalent) to the functor h^X defined by some object X of \mathcal{C} , $h^X(Y) = \text{hom}_{\mathcal{C}}(X, Y)$. Indeed, so it is, and a contract is a legal instrument by which a legal person A undertakes to provide certain goods or services to another person B. But why? The other side of the definition is the *consideration*, or the structure which the values of F carry. A representable functor worth having provides not just sets $F(Y)$, but groups $\text{SL}_2(Y)$, or rings $Y[[t]]$, or some such cattle.

The geometric center of this subject is affine group schemes: representable group-valued functors on commutative k -algebras (k a commutative ring). The abstract center of the subject is arbitrary functors having a left adjoint from one equational category of (universal) algebras to another, and on that high abstract level, all such functors are known. That is, for any algebraic theories \mathbf{S} , \mathbf{T} , the functors $\mathbf{Set}^{\mathbf{S}} \rightarrow \mathbf{Set}^{\mathbf{T}}$ having a left adjoint are represented precisely by the \mathbf{T} -coalgebras in the category of algebras $\mathbf{Set}^{\mathbf{S}}$: a thirty-year-old theorem of P. Freyd [2]. What are the \mathbf{T} -coalgebras? *Explicitly*. Now we are pointing to the center of the subject as an algebraist sees it.

The fundamental result of this work — so described on page 4 — determines the representable abelian group-valued functors on k -rings (rings S with a distinguished homomorphism $k \rightarrow S$); every such functor F is represented by the free k -ring on some (arbitrary) k -bimodule M , the group $F(S)$ being $k\text{-Bimod}(M, S)$. This result is fundamental in much the same way that finding the source of the Nile was the fundamental requirement for European exploration of Africa; when one has gotten that far, what is ahead begins to make sense — we cannot say it is downhill from here, but it begins to look *possible* from here — and on the other hand, the experience of forging past deserts, swamps and hostile natives to the great lakes is a model for the many further journeys that remain. By the way, one of those further journeys vanishes; every representable commutative monoid-valued functor must be group-valued.

The fundamental result is Theorem 13.15, in §13, in Chapter III. Chapter I is a mere 8-page, but inclusive, introduction. Unlike the Nile exploration, the record of predecessors' progress is very short. D. M. Kan [5] determined both the comonoids and the monoids in the category of groups; there are further results in this direction of a very topological tendency, and then one gets to early work of Bergman, joint with W. E. Clark [1], determining a convenient subclass of the co- k -algebra structures on a free cyclic associative k -algebra. Chapter II, three times as long, is the necessary review of coalgebras and representable functors. Then a 20-page run reaches the fundamental Theorem 13.15, near the end of Chapter III. This

1991 *Mathematics Subject Classification*. Primary 16B50, 16W30, 17B99, 18A40, 18D35; Secondary 08B99, 13B99, 14L15, 14L17, 16S10, 16N40, 16W10, 17A99, 20M50.

is in effect the pons asinorum of the book. The proof is not so very formidable, but with four pages mainly about index-strings, with rather intimidating commutative diagrams, and with an empty place in most readers' hearts for the coalgebra we learned at our mothers' knees, it will deter many. The authors anticipate this; they say at the end of Chapter II "... the reader may sometimes wonder, 'What is this all about?' We suggest keeping in mind that what we are studying is simply the possible ways of constructing" reasonably natural associated algebras like $SL_2(Y)$ or $Y[[t]]$. "The concept of the representing coalgebra may be regarded as an elegant bookkeeping device for such constructions, which frees us from particular choices of coordinates for the constructed objects."

Indisputably, Bergman and Hausknecht have done a remarkable job of surveying a broad range of territories, some of which are represented by decisive charts, others by sample products – ivory, apes and peacocks, as it were. Note that the central theorem 13.15 identifies nice *abelian* group-valued functors of k -rings. In the non-abelian ones, there is something like an explosion, surveyed under some restrictions in Theorem 33.7. (Chapter VII. The last chapter is XI: "Directions for further investigation".) Here, let k be a field, cut down from k -rings to k -algebras (in which k is central), and there is an explicit determination of the representable group-valued functors F . It is somewhat indigestible. I shall return to it below, noting here the essential feature, an equivalence between this category of functors and the category of linearly compact k -algebras. (That is, topological k -algebras on an inverse limit of discrete vector spaces. The proof of this theorem is to appear in a forthcoming paper of Hausknecht.)

Chapter VII naturally depends on previous chapters, and we need to notice now Chapter V: "Representable functors from algebras over a field to rings". This chapter determines the representable functors F from (unital) k -algebras, k a field, to a number of categories of not-necessarily-associative-or-unital rings (Lie, anticommutative, commutative, Jordan, ...); the crucial input for Theorem 33.7 is the case that values of F are associative non-unital rings ("rngs"). These functors take Y to $A \hat{\otimes} Y$ where A is a linearly compact vector space with two associative multiplications $*$, $\#$, the range of each lying in the two-sided annihilator ideal of the other, and the multiplication on $A \hat{\otimes} Y$ being the sum of the multiplications of $(A, *) \hat{\otimes} Y$ and $((A, \#) \hat{\otimes} Y)^{\text{op}}$. A sample (a peacock): Let $*$ be any cube-zero multiplication on A (a bilinear operation satisfying $A * (A * A) = (A * A) * A = 0$), and $\# = *$ (or, for an ape, $\# = -*$). This gives a commutative (resp. anticommutative) ring-valued functor. The authors say (p. 120) "it is easy to verify" that these are all the representable functors from unital k -algebras to associative commutative (resp. anticommutative) non-unital rings. They then conclude their investigation into these associative co-rings with several remarks less easy to verify, organized into a page-long exercise.

The Lie ring-valued functors, or Lie co-rings on k -algebras A , are just the Lie (co-) rings underlying, by bracket, associative (co-) rings on A . The authors observe that, in a sense, this is dull; certainly it says, "nothing novel here". But there is something negative and novel; the classical result is just that every Lie ring is *embeddable* in a bracket ring. There is a similar result for unital Jordan co-rings on k -algebras ($\text{char } k \neq 2$); they are "superspecial" Jordan structures, associative structures rewritten with $ab + ba$. In the characteristic-2 case, in contrast, there

are no nontrivial such co-rings. The mere Jordan co-rings (no counit) are characterized, and toward the end of the book there is an extended discussion (pp. 291-294) of generalized Jordan co-rings after N. Jacobson [4] and others.

The various determinations in Chapter V, of all representable ‘blank’-valued functors on k -algebras for a field k (‘blank’ being associative rings with or without 1, Lie, Jordan, and some other rings) are simpler than the general theory would suggest, rather startlingly so. Such functors are always represented by coalgebras, but here, with a small twist, we find them represented by (the far more familiar) algebras. The reason, the authors explain, is the duality between vector spaces and linearly compact vector spaces, which turns comultiplications on a space V into multiplications on the dual space \hat{V} . When k is not a field, no such aid is available. There are nevertheless similar results (found in Chapter VI), not only for k -algebras for a general commutative ring k but for k -rings (k just a unital associative ring). With no dual vector space, how is this possible? Of course, ‘similar’ does not imply ‘equally convenient’. The description of functors with values in various categories of rings involves describing (1) functors with values in arbitrary nonassociative rings, and (2) identities satisfied by values of the functors. Both parts (1) and (2) are rather more difficult if k is not a field. They are carried through by a technique called “element-chasing without elements” using generalized elements, which are just suitable morphisms. Where elements of a dual space \hat{V} were wanted, the generalized elements will be bimodule morphisms $V \rightarrow U$. This is less satisfactory for identities (2) than for bilinear multiplications (1). (The authors note that the generalized elements will be familiar to algebraic geometers. To categorists, too.)

Now, Theorem 33.7. A representable group-valued functor F on k -algebras (k a field) comes, in general, from a general associative, (possibly) non-unital, ring-valued functor $G : Y \mapsto A \hat{\otimes} Y$ by passage to the monoid on $G(Y)$ with multiplication $(s, t) \mapsto s + t + st$ and then to the group of units. The rest of Chapter VII concerns comonoids in k -algebras. The authors describe these structures as “quite diverse, but we develop a class of constructions which may conceivably include them all.” This is not a new note entering in Chapter VII; the longest early chapter, Chapter IV (38 pages), is entitled “Digressions on semigroups, etc.” Sample result from Chapter IV: If V is a representable commutative semigroup-valued functor on k -rings, then each value $V(S)$ has at most one idempotent, and if it has one such, e , then $eV(S)$ is a group.

Chapter VIII is on affine algebra schemes: not the point of the book, but the longest normal chapter, the exception being the last chapter, XI: “Directions for further investigation”. If you want conclusive theorems, avoid both of these chapters. Commutative k -algebras support many more representable group-valued functors than the restrictions of such functors on all (associative unital) k -algebras, beginning among the classical groups. Indeed, GL_n lives on non-commutative rings, represented by a ring with a generic invertible n by n matrix, but SL_n requires commutative support, as do O_n and Sp_{2n} (or perhaps (see Chapter XI) just a distinguished antiautomorphism). A much broader flood of examples rises from the “toy” Boolean ring $B(S)$ in each commutative ring S , composed of the idempotents of S with the natural partial order. B is representable, by $Z \times Z$. Therefore B and Stone duality convert every totally disconnected compact topological algebra (of any pattern \mathbf{T}) into a \mathbf{T} -coalgebra in commutative rings. There is particular interest in the so-called *finite* affine algebra schemes — especially group schemes. A

finite affine scheme (over k) is a representable functor on commutative k -algebras which is represented by a finite-dimensional algebra. The following question seems to remain where J. Tate and F. Oort left it in 1970 [7]: Must a representable group-valued functor F on commutative k -algebras that is represented by a free algebra of finite rank n take values in groups satisfying $x^n = e$? The answer is ‘Yes’ if k is an integral domain, or n is invertible in k , or the values of F are abelian groups. It is known that the values need not satisfy all identities true in n -element groups; in particular, $n = p^2$ does not make all $F(S)$ abelian [7]. Against this background, Bergman and Hausknecht add a number of technical results on values of affine algebra schemes. Sample: If F is an affine monoid scheme over a field, then in every $F(S)$, $xy = e$ implies $yx = e$. Perhaps the best thing in this half of the chapter is a conjecture on the precise form of all affine \mathbf{P} -module schemes F over an algebraically closed field k , where if k has characteristic 0, then \mathbf{P} can be any $\mathbf{Z}/n\mathbf{Z}$, while if $\text{char } k = p$ then \mathbf{P} is $\mathbf{Z}[p^{-1}]$. It is that F is $S \mapsto S^\times$ (group of units) followed by h^G for some G in \mathbf{Ab} which “is” a \mathbf{P} -module. The last half of Chapter VIII is mostly about ring schemes, which cannot be described as easily in the space of a review.

Chapter IX seeks to determine the group-valued, or at least the abelian group-valued, representable functors F on Lie k -algebras. This is done for k a commutative \mathbf{Q} -algebra; the general F is given by any k -module M , and takes each Lie algebra L to the additive group of module homomorphisms of M to L . However, there seems to be no such result for k a field of finite characteristic; at least, a functor F (a co- \mathbf{Ab} object) is described which is *locally*, for each L , isomorphic with the functor given as above by $M = k \times k$ but not naturally equivalent to it. As for non-abelian cogroups in Lie k -algebras, for a field k of characteristic 0 a broad class of them, perhaps all of them, are given by a modification of a construction in §14 of M. Hazewinkel’s book [3].

Chapter X takes the view that ring-valued functors amount to bilinear maps among abelian group-valued functors, and presses on to multilinear maps. This is needed to deal with quadratic Jordan algebras.

There are a few books related to this one. There is no comparable book, or more technically, no book substitutable in significant part for this one. The dedicated reader will soon find herself in love with the very full indexing, which goes well beyond the index to include for each item in the References a list of (from 1 to 24) pages on which it is cited, counting multiplicity. It does the same for items in the eight-page Symbol Index.

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