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Homotopy type and homology, by Hans-Joachim Baues, Oxford Math. Monographs, Clarendon Press, Oxford, 1996, xii + 489 pp., $140.00, ISBN 0 19 851482 4

This is a book which lies somewhat out of the mainstream of present-day research in algebraic topology. In many ways it could have been written thirty or forty years ago. To place it in its proper context, it seems best to begin this review with a historical sketch.

Combinatorial topology was very largely invented by Poincaré towards the end of the nineteenth century (of course at that period topology was generally known as analysis situs). The idea was to study topological spaces, such as manifolds, not as sets of points, but as collections of cells, for example simplexes. The topology of the cells was regarded as well understood; the interest resided in the way they fitted together. Although algebraic topology evolved out of combinatorial topology, it cannot simply be said to have replaced it; the situation is more complicated than that.

For one thing the process of algebraization was a gradual one. When Veblen gave his Colloquium lectures in 1916, he, like Poincaré, saw homology in terms of numerical invariants, the Betti numbers and torsion coefficients. It was not until ten years later that it began to be usual to see homology in terms of abelian groups. The advantages of doing so can be appreciated by comparing the original proof of the invariance of homology by Veblen's protegé Alexander with a modern version which, while not essentially different, is far more algebraic and much shorter.

In the post-Poincaré era the leading topologist was Brouwer, who, in two wonderful years, achieved one breakthrough after another. Simplicial approximation was one of his ideas, the essential tool in the proof of the invariance of homology. Another was the notion of degree. Later, although Brouwer himself was not directly active in topological research, he exercised a strong influence on the development of the subject through the young people who came from all over Europe to work with him—people such as Aleksandroff, Freudenthal, Hopf, Hurewicz and Vietoris.

It was in Brouwer's circle that homotopy theory developed. The term "homology" is due to Poincaré, but the term "homotopy" first appears in the 1907 *Enzyklopädie* article of Dehn and Heegard. Brouwer himself, and then Hopf, made a start in the problem of classifying maps of one sphere into another by homotopy. If the spheres have the same dimension, the degree is all that is needed. If the domain sphere has lower dimension, all the maps are homotopic; but if the domain has higher dimension, the situation is quite different. Even today it is only partially understood.

It was not until 1935 that progress began to be made in tackling questions of this kind. Although Hopf also played a major role, it was above all Hurewicz who laid the foundations of homotopy theory in the "research notes" he published in 1935-36. Among many other ideas and results, he showed, in the theorem which bears his name, how homology could be used to help classify maps. At the same time he introduced the notion of homotopy type, a less rigid way to classify spaces

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than the classification by homeomorphism. In the case of manifolds, for example, classification by homotopy type is an important first step in classification by homeomorphism or diffeomorphism. Moreover this is by no means the only way in which homotopy-theoretic ideas enter into the modern theory of manifolds.

Although homotopy theory by-passed some of the deep problems of topology proper, it turned out to have plenty of problems of its own. Although many topological problems can be reduced to problems in homotopy theory, that did not mean they could be regarded as completely solved. So researchers started to look for "first approximations" to homotopy theory, where these difficulties might be overcome.

Stable homotopy theory is one of the most important of these first approximations. It originated in the work of Freudenthal, another member of Brouwer's circle, but it was Spanier and (J. H. C.) Whitehead who realized the full potential of the idea. In recent years a large part of research in homotopy theory has been on the stable theory, and at least its architecture is beginning to be quite well understood. However, progress has also been made in other directions.

Localization has been a common procedure in algebraic geometry for many years. The corresponding procedure in algebraic topology was developed by Serre, Adams and others. Rationalization is a particularly manageable form of localization, as shown by Quillen, and rational homotopy theory provides another first approximation, which has also attracted much research interest. In particular the minimal models of Sullivan reduce the rational theory to a simple form, although the simplicity is deceptive. Other forms of localization are also part of the standard practice of modern homotopy theory.

Progress has also been made in other areas of the theory, including the one which is the subject of the book under review. Baues quotes from a lecture Whitehead gave at the 1950 International Congress: "The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry." Whitehead devoted several years to constructing algebraic models of the homotopy types of certain classes of spaces. Although this program was continued a little further by Chang Su-Chen, a former research student of Whitehead's, interest in it seemed to have lapsed. One explanation for this might be that the viewpoint which Whitehead described as "combinatorial homotopy" was in some ways superceded by the dual approach of "Postnikov systems".

In the last few years, however, Baues has taken up Whitehead's program again and made substantial progress. The algebraic models he constructs are complicated, but they do provide the answers to questions such as: how many different homotopy types of simply connected five-dimensional complexes are there, and how can you tell them apart?

The book under review gives a reasonably self-contained account of this work, some of which has already been published in the form of articles. Prerequisites, he suggests, are elementary topology, elementary algebra and some basic notions of category theory. Although Baues is nothing if not thorough, I feel myself that a reader without a reasonable grounding in ordinary homotopy theory would find it heavy going.

Progress in mathematical research is often achieved by redefining the objectives. However, it is good to look back from time to time and think again about questions which interested mathematicians in the past. Baues has performed a valuable service by doing this in the case of the Whitehead program. This is not a book which is likely to have wide appeal, but I would hope to find it in the library of any institution where topology is taken seriously.

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