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Positive harmonic functions and diffusion: An integrated analytic and probabilistic approach, by Ross G. Pinsky, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995, xvi + 474 pp., vol. 45, \$80.00, ISBN 0-521-47014-5

Consider the second order, elliptic differential operator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\,\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + V(x)$$

where a_{ij}, b_i, V are nice functions in $\mathbb{R}^d$ (say Lipschitz). Fixing a domain $D \subset \mathbb{R}^d$ we denote by $C_L(D)$ the class of all positive solutions of the equation $Lu = 0$ in D. Properties of $C_L(D)$ may be studied from an analytic point of view, a probabilistic one using diffusions, or with techniques from both fields combined. The point of the book under review is to give an exposition of this last, integrated approach to the subject.

The analytic approach has a longer tradition. For example, the maximum principle dates back to Gauss in 1838. The probabilistic approach is a product of this century. Undoubtedly, credit for the foundational step should go to Wiener [19]. This step is the construction of the measure P on the path space $\Omega = \{\omega \in C([0,\infty),\mathbb{R}^d) : \omega(0) = 0\}$ representing the trajectories of particles undergoing Brownian motion. The function $W : [0,\infty) \times \Omega \to \mathbb{R}^d$ defined by $W_t(\omega) = \omega(t)$ has the distributional property that under P the random variables, $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent, Gaussian mean 0, with variances $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$. It was known almost from the beginning that P gives mass one to paths which are nowhere differentiable.

The next, audacious step is due to Itô [10], who gave a representation of the diffusion naturally associated to the operator $L_0 = L - V$. The association between the diffusion X and operator L_0 is captured by the requirement that as $h \to 0$,

$$\begin{aligned} E(X^i_{t+h} - X^i_t \mid \sigma(W_s : s \le t)) &= b_i(X_t)h + o(h), \quad 1 \le i \le d\,, \\ E((X^i_{t+h} - X^i_t)(X^j_{t+h} - X^j_t) \mid \sigma(W_s : s \le t)) &= a_{ij}(X_t)h + o(h)\,, \quad 1 \le i,j \le d\,, \end{aligned}$$

where E denotes integration with respect to P and $E(\cdot|\sigma(W_s : s \le t))$ denotes conditional expectation with respect to the σ- field $\sigma(W_s : s \le t)$. By the above distributional properties of W it is natural to propose, as Paul Lévy did, that X ought to solve the differential equation

$$\begin{aligned} dX_t(x) &= \sigma(X_t(x))dW_t + b(X_t(x))dt \\ X_0(x) &= x\,, \end{aligned} \tag{0.1}$$

where σ is any square root of $a, \sigma\sigma^* = a$. The difficulty, of course, is that with P probability one W does not have a derivative anywhere. However, by using stochastic integrals, Itô was able to establish existence and uniqueness of the solution to (0.1) under Lipschitz conditions on σ and b. These smoothness requirements have

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been relaxed using the "martingale problem" approach. (See the book of Stroock and Varadhran [18] for details.)

This representation (0.1) provides considerable leverage. First, it yields an intuition for the dynamics of X. In integrated form,

$$X_t(x) = x + \int_0^t \sigma(X_s(x))dW_s + \int_0^t b(X_s(x))ds \tag{0.2}$$

exhibits X as a perturbation of the dynamical system $\dot{\varphi}_t = b(\varphi_t)$. The perturbing term, the stochastic integral of $\sigma(X)$ with respect to W, is an extremely well understood object.

Often information obtained from (0.1) can be translated into fairly deep results. Knowledge of just where the paths X may reach can be turned into a maximum principle for L_0 as was done by Stroock and Varadhan [17]. Fairly refined information about harmonic measure can be deduced as for the case $\sigma(x) = \epsilon$ in the so-called "exit problem" treated by Freidlin and Ventcel [9]. In this problem, the flow φ corresponding to b goes into D in the sense that $b \cdot \vec{n} > 0$, $\vec{n}$ = inward normal, at all points of ∂D. If $X_t^\epsilon(x)$ is the solution of (0.2) with $\sigma(x) \equiv \sqrt{\epsilon}$, $X_t^\epsilon(x)$ will more and more closely follow φ as ϵ tends to zero. Yet $X_t^\epsilon(x)$ will, through persistence, still exit D even when $x \in D$. Using large deviation information about Brownian motion, Freidlin and Ventcel [9] were able to identify the limiting exit distribution of $X^\epsilon(x)$ from D as ϵ tends to 0. Characterization of bounded L_0-harmonic functions or even $C_{L_0}(D)$ can be gotten by careful examination of the exit behavior of X from D as in Cranston [4], [5]. These are, of course, only a few of many examples.

Second, the representation (0.1) provides a firm link between X and L_0 as furnished by Itô's formula:

$$u(X_t) - u(x) - \int_0^t L_0 u(X_s)ds = \int_0^t \langle \nabla u(X_s), \sigma(X_s)dW_s\rangle \ , \tag{0.3}$$

for $u \in C^2(\mathbb{R}^d)$. This offers the opportunity to 'measure' X by taking functions of this process. The right-hand side in (0.3) is a local martingale (analog of the sequence of fortunes of a gambler in a fair game) with respect to the σ-fields $\mathcal{F}_t = \sigma(W_s : s \le t)$. Since local martingales are constant on average and the right side in (0.3) is zero at $t = 0$, (0.3) has the additional interpretation that X is in some sense the dynamical system determined by the "vector field" L_0. Further, (0.3) is especially fruitful when applied to $u \in C_{L_0}(D)$.

Third, uniqueness of solutions to (0.1) implies the strong Markov property. To state this, we need to introduce

- stopping times: $\tau : \Omega \to [0, \infty)$ is a stopping time if $\{\tau \le t\} \in \mathcal{F}_t, \quad \forall t \ge 0$.
- the σ-field of events up to time $\tau : \mathcal{F}_\tau = \{\Lambda \in \mathcal{F}_\infty\}$ such that $\Lambda \cap \{\tau \le t\} \in \mathcal{F}_t\,, \quad \forall t \ge 0$.
- the shift operator: $\theta_t : \Omega \to \Omega,\ \theta_t\omega(\cdot) = \omega(t + \cdot)$.

Then for f bounded measurable, the strong Markov property is

$$E[f(X_{t+\tau})|\mathcal{F}_\tau] = E[f(X_t \circ \theta_\tau)|X_\tau] \ . \tag{0.4}$$

Intuitively, this means that X will forget its past at such times τ and proceed as if it were starting at time 0 from the position X_τ.

Finally, a weight can be given to paths up to time t, $\exp\{\int_0^t V(X_s)ds\}$, in order to make direct statements about L using X. This involves the Feynman-Kac formula, which may be stated as:

$$(0.5) \qquad u(X_t)\exp\left\{\int_u^t V(X_s)ds\right\} - u(x) - \int_0^t \exp\left\{\int_0^s V(X_r)dr\right\} Lu(X_s)ds$$

is a local martingale for $u \in C^2(\mathbb{R}^d)$.

Much can be said about L and $C_L(D)$ using (0.1),(0.2),(0.3),(0.4),(0.5).

The focus of the present book is to apply these ideas together with analytic techniques to a trichotomy for the operator L:

(A): L has a Green function on D (L is subcritical).
(B): L has no Green function on D yet $C_L(D) \neq \phi$ (L is critical).
(C): $C_L(D) = \phi$ (L is supercritical).

Probabilists might recognize the distinction, in the case $V \equiv 0$, between (A) and ((B) or (C)) as the dichotomy between transience and recurrence for the process X. These ideas concerning criticality have arisen in various contexts in recent years. Simon [15] has applied criticality in the study of L^p properties of Schrödinger semigroups. Nachman [13] used it in his solution of the two-dimensional Calderon problem. Zhao [20] established the equivalence of subcriticality and finiteness of the so-called gauge, $E\left(\exp\left\{\int_0^{\tau_D} V(X_s(x))ds\right\}\right), x \in D$, where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ and X is the diffusion associated to L_0.

In the subcritical or critical case, one is immediately confronted with the task of characterizing $C_L(D)$. The critical case is quickly handled by the trick of using the ground state transform. For $h \in C_L(D)$, the operator $L^h = \frac{1}{h}L(h\cdot)$ has constant solutions if and only if $C_L(D)$ is one dimensional. It's also easy to see L has a Green function if and only if L^h does. Since L doesn't, L^h (which has no zeroth order term) is associated to a recurrent diffusion. A simple classical argument shows that an operator associated to a recurrent diffusion can support no nonconstant positive solutions (this uses (0.3) and the martingale convergence theorem). The subcritical case is subtler and reveals a fascinating interplay between L, the geometry of D, and the asymptotic (exit) behavior of X as it approaches ∂D. This interplay is captured by the Martin boundary introduced in Martin's classic paper [11]. This boundary is obtained by adding just enough idealized boundary points to make the ratio of Green functions

$$K_y(\cdot) \equiv \frac{G(\cdot, y)}{G(x_0, y)}, \qquad x_0 \in D \text{ fixed},$$

continuous on compact subsets of D as y goes out to ∂D. The wonderful thing about Martin boundary theory is that K_{y_n} need not converge even if y_n converges to a point in Euclidean space. (The reader should try $D = \mathbb{C} - [-1, 1]$ and $y_n = (-1)^n i/n$ and $L = \frac{1}{2}\Delta$.) Not all Martin boundary points have the same status; some are known as minimal. A Martin boundary point ξ is called minimal if K_ξ is a minimal element of $C_L(D)$ in the sense that if $h \in C_L(D)$ and $0 < h \leq K_\xi$, then $h = CK_\xi$ for some constant C. Martin's representation theorem asserts that to every $h \in C_L(D)$ there is a unique positive Borel measure μ, concentrated on the minimal Martin boundary points such that

$$h(x) = \int K_\xi(x)\mu(d\xi) \ .$$

Of course, this is more meaningful in particular instances if the Martin boundary can be described in some fairly explicit manner. From the probabilistic point of view, Doob [8] first drew the connection between X and $C_{L_0}(D)$ by showing that $\lim_{t\uparrow\tau_D} X_t$ exists in the Martin topology and is with probability one a minimal Martin boundary point. It's interesting to note that he made use of the above mentioned ground state transformation for $h \in C_{L_0}(D)$ (actually Doob treated only $L_0 = \frac{1}{2}\Delta$) and called the process corresponding to L^h the h-transform of Brownian motion, denoted by X^h. When $h = K_\xi$ and ξ is minimal, Doob showed that X^h converges to ξ in the Martin topology as X^h exits D. The purpose of his study was to obtain a probabilistic version of Fatou's boundary limit theorem for harmonic functions. This paper has been very influential and a large body of results have been produced in the '80's and '90's using h transforms. The reader is referred to the book of Chung and Zhao [3] or the article of Bañuelos [2] for more information and references. The convergence result of Doob can be used to give reasonable conjectures about the Martin boundary. For example, in Cartan-Hadamard manifolds with sectional curvatures pinched between two negative constants, Prat [14] showed that while Brownian motion drifts off to infinity, its geodesic spherical coordinate will converge. Since it must be converging in the Martin topology, this means the Martin boundary should be a sphere at infinity. This was proved by Anderson and Schoen [1] by analytic means.

Finally, the basic trichotomy is very interesting when applied to the operators $L - \lambda$ for $\lambda \in \mathbb{R}$. Here either $L - \lambda$ is supercritical for all $\lambda \in \mathbb{R}$ or there is a critical value λ_c for which $L - \lambda$ is supercritical for $\lambda < \lambda_c$ and subcritical or critical for $\lambda = \lambda_c$ and subcritical for $\lambda > \lambda_c$. In the case of bounded domains, λ_c is the top of the spectrum of L, call it λ_0, whose existence is guaranteed by 'Fredholm theory' since L^{-1} is a compact operator. For unbounded domains this compactness of L^{-1} may not be true, but there are still ample reasons to think of λ_c as a generalized principal eigenvalue. The diffusion corresponding to L_0 provides a convincing connection between λ_0 and λ_c due to Donsker and Varadhan [7] for λ_0 and Stroock [16] for λ_c:

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} E\left[\exp\left\{\int_0^t V(X_s(x))ds\right\}; \tau_D > t\right], \quad D \quad \text{bounded},$$

$$\lambda_c = \sup_{A\subset\subset D} \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} E\left[\exp\left\{\int_0^t V(X_s(x))ds\right\}; \tau_A > t\right], \quad D \quad \text{unbounded}.$$

Thus, if D is bounded, λ_0 is the Feynman-Kac weighted exponential rate of escape of X from D; while if D is unbounded, λ_c is the Feynman-Kac weighted exponential rate of escape from compact subsets of D.

This book occupies a unique niche in the literature by centering its attention on the criticality trichotomy and treating rather general elliptic operators. It contains numerous interesting examples which well illustrate the central ideas. As it also contains many exercises and is fairly self-contained it would serve well as a graduate text for a topics course. Researchers in the field, whether analyst or probabilist, should find this an informative book. This reviewer enjoyed it. The first edition contains periodic misprints. A few significant references are missing such as Zhao [20], deBlassie [6], Meyers and Serrin [12]. However, these are only minor quibbles, and the quality of the material and the clear exposition more than outweigh them to make this book a valuable contribution.

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