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Super-real fields—Totally ordered fields with additional structure, by H. Garth Dales and W. Hugh Woodin, Clarendon Press, Oxford, 1996, 357+xiii pp., $75.00, ISBN 0-19-853643-7

1. Automatic continuity

The structure of the prime ideals in the ring $C(X)$ of real-valued continuous functions on a topological space X has been studied intensively, notably in the early work of Kohls [6] and in the classic monograph by Gilman and Jerison [4]. One can come to this in any number of ways, but one of the more striking motivations for this type of investigation comes from Kaplansky's study of algebra norms on $C(X)$, which he rephrased as the problem of "automatic continuity" of homomorphisms; this will be recalled below. The present book is concerned with some new and very substantial ideas relating to the classification of these prime ideals and the associated rings, which bear on Kaplansky's problem, though the precise connection of this work with Kaplansky's problem remains largely at the level of conjecture, and as a result it may appeal more to those already in possession of some of the technology to make further progress – notably, set theorists and set theoretic topologists and perhaps a certain breed of model theorist – than to the audience of analysts toward whom it appears to be aimed. This book does not supplant either [4] or [2] (etc.); it follows up on them.

Kaplansky pointed out the significance of the issue of "automatic continuity" in [5] in the following form: if $C(X, \mathbb{C})$ is the Banach algebra of continuous complex valued functions on a compact topological space X, under the uniform norm $|\cdot|_X$, is every homomorphism from this algebra into a Banach algebra continuous? Bade and Curtis gave a very perspicuous analysis of the content of this question [1], which by subsequent work of Sinclair [7] can be recast in the following form:

> Is there a non-maximal prime ideal P in the ring $C(X)$ (i.e., $C(X, \mathbb{R})$) for which the quotient ring $C(X)/P$ is *normable* (carries an $\mathbb{R}$-algebra norm)?

Note that for any such prime ideal P the natural map $C(X) \to C(X)/P$ furnishes a discontinuous homomorphism, since P is dense in the maximal ideal containing it.

After this phase of fairly radical reformulation, Kaplansky's original question was eventually answered. To quote from the present book: "Constructions of discontinuous homomorphisms from $C(\Omega, \mathbb{C})$ for each infinite compact space Ω were given by Dales (1979) and by Esterle (1978). ... The proof that ... all homomorphisms from each algebra $C(\Omega, \mathbb{C})$ are continuous was first given by Solovay in 1976, using a condition of Woodin."

This no doubt seems a bit odd. The last ellipsis in the above hides the phrase "it is relatively consistent with [set theory] that"; with that emendation it still seems a bit odd. However the "proofs" of Esterle and Dales referred to are also

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consistency proofs, showing in fact that the continuum hypothesis is sufficient to settle Kaplansky's question.

This striking result is one of a large heap of examples showing that a very wide variety of mathematical questions, involving some dose of abstract (or "modern") ideas, fall with surprising regularity into the domain of the set theorists, in the sense that if one insists on a definite solution, then one is going to have to make explicit some additional set theoretical hypothesis. The path from norms on $C(X)$ through automatic continuity to set theoretic issues is presented with elegance in Chapters 1 and 3 of the first book by this pair of authors [2], and in a number of further references given in that volume.

2. The effect of CH

As stated, the continuum hypothesis (CH) gives rise to algebra norms on quotient rings $C(X)/P$, with X compact and P prime. In fact the continuum hypothesis enters into the argument only in the following very limited way (but the parallel argument, showing that in the *absence* of CH such norms may not exist, is thoroughly set theoretic in character [2]). There are two main facts:

F1: For any prime ideal P in $C(X)$, the field of fractions K of the integral domain $C(X)/P$ is a real-closed field, and $C(X)/P$ is contained in the subring K^{fin} of *finite* elements of K (i.e., the smallest convex subring containing 1).

F2: If K is a real closed field, and an $\mathbb{R}$-algebra, of cardinality at most $\aleph_1$ (or even of transcendence degree at most $\aleph_1$ over $\mathbb{R}$), then the subring K^{fin} is normable.

These are both true without any special assumptions. Furthermore, there are numerous examples of rings of the form $C(X)/P$ of cardinality at most $2^{\aleph_0}$ (including most of the examples that first come to mind). If we invoke the continuum hypothesis at this stage, we see that these facts solve Kaplansky's problem, as many of the rings mentioned in (F1), with P nonmaximal (as required to settle Kaplansky's problem), fall under (F2).

The first fact is part of the basic setup as one would understand it on the basis of [4]. The second is a major result proved for the particular purpose of settling Kaplansky's problem. The present book was motivated originally by the desire to have better results of the type of (F2); but this led back to the problem of understanding more fully what sorts of fields K really occur in (F1), and remained there, as the situation is rather complex.

Let us consider (F1) in more detail, in the context of an example. Let $X = \mathbb{N}^*$ be the one-point compactification of $\mathbb{N}$, that is, the sequence of the natural numbers with one point at infinity. Then $c = C(\mathbb{N}^*)$ is essentially the ring of convergent sequences of real numbers. The maximal ideals are the kernels of the evaluation maps $f \mapsto f(x)$ for $x \in \mathbb{N}^*$; in terms of sequences, these are the homomorphisms λ_i taking the sequence (r_n) to r_i, together with λ_∞ taking (r_n) to $\lim r_n$. The prime ideals can easily be seen to consist of the kernel of λ_n for $n < \infty$, plus a large number of prime ideals P contained in the kernel c_0 of λ_∞ (c_0 is of course the ideal of sequences which converge to 0). To understand the prime ideals lying in c_0, one begins with the minimal primes: these turn out to be classified by *ultrafilters* $\mathcal{U}$ on $\mathbb{N}$. For each ultrafilter $\mathcal{U}$, one defines $P_\mathcal{U}$ as the set of sequences (r_n) such that the set:

$$\{n : r_n = 0\}$$

lies in the ultrafilter $\mathcal{U}$. As it would be difficult to impose a more stringent condition on a function, one is not surprised that these are minimal primes; to see that every prime contained in c_0 contains one of these, one notes, mainly, that if $A \cup B = \mathbb{N}$, and if f_A, f_B are continuous functions vanishing on A and B respectively, then $f_A f_B = 0$ and hence at least one of these functions is in P.

The picture that emerges then is this: all of the prime ideals we have explicitly identified (namely, the maximal and the minimal ones) are determined by specifying at what points a function is required to vanish. Such ideals are called *z-ideals*, a very important class indeed, and a somewhat odd sounding name, referring to the fact that they are determined by prime filters of closed sets and that the only closed sets of any actual relevance are the so-called *z-sets*, which are those which actually occur as the set of zeroes of some continuous function. The other prime ideals lie somewhere between a minimal prime P^- and a maximal prime P^+, and the interval from P^- to P^+ is linearly ordered. This picture passes over to general compact X as well, with the proviso that many more z-ideals may turn up in addition to the minimal and maximal ones. The other prime ideals may be thought of as determined by "growth rate" conditions rather than "vanishing" conditions, and they have a considerably more subtle character. Many of the complexities of the present book relate to getting control of the general prime ideal comparable to what one can achieve in the z-ideal case.

Returning to the minimal prime ideals $P = P_{\mathcal{U}}$ below c_0 in c, classified by ultrafilters $\mathcal{U}$, we may ask about the structure of the ring $C(\mathbb{N}^*)/P$ and its field of fractions; these will be denoted A_P and K_P respectively. K_P can easily be identified with the model of *nonstandard analysis* furnished by the ultrapower $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$. The finite part K_P^{fin} corresponds to $l^\infty/\mathcal{U}$, the ring of bounded sequences modulo $\mathcal{U}$, while A_P is the subring corresponding to sequences which converge on *some* set in $\mathcal{U}$. Note that any bounded sequence has a well-defined *limit with respect to* $\mathcal{U}$, to which it converges in the sense that it gets ϵ-close to the limit on a set $A_\epsilon \in \mathcal{U}$; this is a weaker condition than *ordinary convergence* on one set in $\mathcal{U}$. It does however occur that for particular ultrafilters $\mathcal{U}$, these two conditions are equivalent; thus it may happen that $A_P = K_P^{\text{fin}}$, in which case the associated ultrafilter is called a P-point (with an unfortunate but ephemeral clash of notation with the prime P), or that $A_P \neq K_P^{\text{fin}}$ – though set theory intervenes again, as the existence of P-points, while following from CH, cannot be proved without some such extraneous hypothesis. In other words, the situation is already quite tricky here.

As K_P is constructed from A_P, such questions as whether $A_P = K_P^{\text{fin}}$ reflect intrinsic properties of A_P, and hence of P itself. On the other hand, the field K_P *appears* to contain much less information: it is a real-closed field, and in the cases mentioned so far it is also an η_1-*field*; that is: if A, B are two countable subsets of K_P, with $A < B$, then they are separated by an element of K_P (the subscript '1' stands for: $< \aleph_1$, i.e.: countable), and this is typical though not universally valid. The key question for the present book would be the following: if we jettison A_P and keep only K_P, how much information can we decode about P? This being the key to the book, we will hold off for a moment before plunging into it, and lay out a little more of the book's terminology.

3. Hyperreal and superreal

When M is a maximal ideal in a ring $C(X)$, the quotient field $C(X)/M$ is both A_M and K_M in the notation used above. For X compact, this is just another copy of $\mathbb{R}$, but for X noncompact, proper extensions of $\mathbb{R}$ – which are real-closed fields, and are η_1 – result; they come with a built-in $\mathbb{R}$-algebra structure, coming from the image of the constant functions. These fields are often referred to as *hyperreal* fields. The authors propose to call the fields K_P associated to prime ideals generally the "superreal" fields – again, omitting the case of $\mathbb{R}$ itself.

A fact worth highlighting, mentioned in passing on p. 82 of the present book but taken for granted a few pages earlier, is that every superreal field K_P associated with a prime ideal P of X is also of the form $K_{P'}$ for some prime ideal P' in $C(X')$ with X' compact (and, indeed, X' can be the Stone-Čech compactification of X). The noncompact case is important if one thinks only in terms of maximal ideals and quotient fields rather than in terms of prime ideals, quotient rings, and fields of fractions, as the "reduction" to the compact case then becomes irrelevant, but in all other cases that actually arise one may assume at the outset that X is compact; in particular, it is convenient that in this case one has $C(X)/P \leq K_P^{\text{fin}}$.

For Kaplansky's problem, it was important to understand whether $C(X)/P$ could be normable, with P prime and nonmaximal and X compact. For the results under CH, where these rings often are normable, the reason for this was that K_P^{fin} itself was normable. Following on the ambivalent solution of the automatic continuity problem, the authors set themselves the task of elucidating the situation further under the assumption of CH, the main question being phrased something like this (p. 108, approximately Question 2):

$$\text{When is } K_P^{\text{fin}} \text{ normable?}$$

(The authors would write $K_P^{\#}$ here – apart from this I am generally following their notation.)

The other version of this question, also given in Question 2, p. 108, would have $A_P = C(X)/P$ in place of K_P^{fin}, and would be closer to the Kaplansky problem; in any case the development drifts away fairly rapidly from both forms of Question 2. In fact the state of affairs is rather more complicated by the end of the story than it was at the outset. Initially, there seems to be some possibility of getting an exact classification of the fields K_P for which K_P^{fin} is normable (perhaps even fairly cheaply), under the simplifying assumption CH, which is in force here whenever it is of any use. Certain complications begin to arise, to which we will return, on p. 112 (condition 5.2), for a class of fields which *a priori* might even be empty; once it is seen that there are, in fact, such fields, the possibility of a cheap solution begins to recede. This then leads the authors to attempt a more systematic analysis of superreal fields in general, and here I quote (p. 77): " ... we began our study with the intention of characterizing the superreal fields algebraically in the class of all totally ordered fields, but the possibility that there is an intrinsic characterization of these fields disappeared over the horizon ... ". In view of this, it is rather hard to gauge at this stage to what degree we have learned anything about Kaplansky's problem. My understanding is that the main hope was that the relevant fields K_P would be similar enough that an approach that may be characterized loosely as "norm one, you've normed them all" would work. In any case, what the book supplies in ample measure is a new set of tools for distinguishing very similar looking real closed fields, or on occasion (with the same tools) showing they are isomorphic.

(A digression: The details seem vaguely reminiscent both of Shelah's nonstructure arguments and of work on recursively saturated models of arithmetic; both are probably relevant, the latter more immediately than the former. These phenomena may not be entirely welcome, but they are real, and they open up some possibilities that could probably be profitably pursued by set theorists, set theoretic topologists, and the occasional model theorist. Now, back to our story.)

The first question would probably be whether every superreal field is hyperreal, which can be answered negatively and cheaply on the basis that hyperreal fields are automatically η_1 and superreal fields are not, but the book is in fact about the superreal fields which *are* η_1. With this additional constraint one can separate these two classes "consistently", that is in a model of set theory built by forcing (Chapter 11). An almost equally basic question, which however goes unanswered and indeed occurs in a more refined form as the last and "most interesting" (I certainly agree) of the list of open questions in Chapter 12, is whether z-ideal η_1 fields (that is, $\mathbb{R}$-algebras which occur as K_P for P a z-ideal and which are η_1) are in fact hyperreal (that is, whether the same $\mathbb{R}$-algebra, up to isomorphism, also occurs as a hyperreal field, $C(X)/M$, where X of course would then be noncompact).

By the end of the story one has in fact wondered, with varying degrees of success, about the relationships among quite a variety of properties of superreal (and, let us suppose, also η_1) fields, namely: z-ideal, exponential, valuation, strongly convex, cut prime, hyperreal, ultrapower – not to mention, say, strongly convex z-ideal, etc. What we have here are mostly some rather natural properties of prime ideals in $C(X)$, transferred to superreal fields by requiring that the field in question be realizable as K_P where the prime ideal P has the property in question. Since decoding any information about P from K_P is a tricky proposition, one has at the outset the possibility that any field that can be realized in one way can be realized in all ways, and one has to find some way of distinguishing these types of fields intrinsically.

4. Vive la différance

An example is in order. One finds on three occasions as one proceeds through the book (namely, p. 104, p. 248, and p. 338) successive incarnations of an increasingly baroque chart which reflects in highly simplified form the various relationships among the classes of superreal fields under consideration, with a charming legend whose two shades of grey may be mystifying on their first unanticipated appearance on p. 248, especially in dim light, in which case I recommend sunlight or a glance at p. 338.

According to these charts, one has learned that real closed η_1-fields are not necessarily superreal by p. 248, and indeed this is a relatively early result, Theorem 6.23 on p. 140. The proof goes as follows: there is an "operational calculus" on any superreal field, not too surprisingly in terms of their origin: this means that one can extend suitably nice (Lipshitz) functions naturally to act on the "unit interval" $[0,1]_{(P)}$ of any K_P, with K prime, and this leads to a definition of a "partial exponential" map exp from a convex subgroup of $(K_P,+)$ into $K_P^{>0}$, though not necessarily onto. Even this much is not possible in an arbitrary real closed field, as one shows using a rather large field constructed as the full Hahn power series field (generalized Puiseux series with well-ordered support, in an ordered divisible group of exponents) where the group of exponents is itself a *restricted* Hahn power series group (with countable support). This is one of the sorts of "intrinsic differences" which will play a role in the development. Others are gap structure and induced models of arithmetic and analysis. The gap structure simply relates to the obvious invariants of gaps (unfilled Dedekind cuts) in ordered sets: the cofinality (from below) and coinitiality (from above) of the gap. The induced models of arithmetic and analysis are something quite new and give rise to the subtitle of this volume: *Totally ordered fields with additional structure*.

5. Additional structure

Since the elements of any of the fields K_P of interest are represented by quotients of continuous functions (with sufficiently sparsely vanishing denominators, more or less), one can ask which of these elements are represented by integer-valued functions. With the question phrased this baldly, one is tempted to answer, in general, "precious few," and move on; but if one recalls that working modulo P means, in particular, working modulo a z-ideal below P and that consequently we need only work with functions defined on some member of an associated prime z-filter, then it is not surprising that, on the contrary, something like a nonstandard model of arithmetic emerges. Indeed, if one works with a z-ideal, one gets a limit of ultrapowers which is cofinal in the field K_P, and one may go on to look at reals (taken as binary sequences of $0, 1$) coded in the model by elements of, say, $[0,1]_{(P)}$. For general ideals the resulting structure is going to be considerably messier and is really just the image in K_P of part of the model associated with a z-ideal below P; but then this turns out not to be closed under exponentiation in general, and it turns out to be technically superior to restrict attention to the integers n for which 2^n does in fact exist – which alters nothing in the z-ideal case. In any case one has here a potential "invariant" of the field K_P, with the proviso that it is initially defined from the ideal P, and consequently if several such fields are isomorphic, they may carry several quite different models of arithmetic.

This arithmetical structure leads, for example, to a better example of a real closed η_1 field which is not superreal: in this example, we find a real closed η_1 field whose *order type* cannot be the order type of a superreal field. One considers both gaps and the induced (fragmentary) "model of arithmetic" in a superreal field, as follows.

One can use straightforward restricted and unrestricted Hahn power series constructions to produce real closed η_1 fields with various types of gaps in them: in particular, we can produce one such field with one gap of type $(\aleph_0, \aleph_1)$ and another of type $(\aleph_0, \aleph_2)$ (we give the cofinality/coinitiality invariant here). But a striking result (Theorem 8.61, p. 198) of the present work states that for K_P associated with a prime, nonmaximal ideal in a compact space, with K_P an η_1 field (a little less will do), any gap of type $(\aleph_0, \lambda)$ must have λ exactly the cofinality of the set of infinitesimal elements of K_P (the maximal ideal of K_P^{fin}). This involves a bit of a journey: the original cut is moved into the interval $[0,1]_{(P)}$ (fair enough), and assuming that λ is uncountable, it then gives rise to a similar gap in the "encoded" binary sequences of suitable (nonstandard) length, which turns into a gap in the "nonstandard integers"; there, doing a bit of arithmetic, one discovers that λ must be the coinitiality of the set $\mathbb{N}_{(P)} \setminus \mathbb{N}$ (the infinite nonstandard integers). This is what one really wants, namely a value for λ, but then it translates back into something that makes sense intrinsically in K_P.

I have studiously avoided discussing the rather formidable core of this work in any detail and have gone into far more detail than I would normally do concerning the basic set-up, because this is a book which has something of the character of a couple of advanced articles feeling their way into new territory, and one has to work to an unusual degree at keeping the picture in focus.

6. The book

However, it is now time to say how this book is in fact organized.

The first four chapters provide an efficient introduction to the subject that readers will find convenient and on occasion indispensable, though most of the material is standard. One will probably begin with Chapter 5 or a quick scan of the preceding chapters. In Chapter 5 one will find a discussion of Kaplansky's problem and normability issues, including a number of results which lead to the key fact that under the generalized continuum hypothesis (say) one knows in most cases which ordered fields K have K^{fin} normable; but in one case, when K has cardinality $\aleph_2$ and its value group (relative to K^{fin}) has cardinality $\aleph_1$ (this is condition 5.2, p. 112), the question has not been settled either way.

Since we are wondering whether there are any, let us call superreal fields meeting this pair of conditions *extraterrestrials* for the moment.

From this point on the authors begin to worry more about issues like: are there any extraterrestrials (and of which types, exactly), rather than normability *per se*. Let me stress that CH and GCH are invoked wherever needed. The theorems are very clearly marked with respect to such hypotheses, but some of the discussion in the book does not harp on the point, nor will I below.

With this, the book begins. Chapter 6 deals with the operational calculus in general and exponentiation in particular and whether three explicitly constructed Hahn power series fields are in the classes of concern to us. These fields, denoted $\mathcal{R}$, $\tilde{\mathcal{R}}$, and $\hat{\mathcal{R}}$, play a large role in this book (for example $\hat{\mathcal{R}}$ is the one called on in Theorem 6.23, described above), but this cannot be conveniently enlarged on here, though I should add that while $\mathcal{R}$ and $\hat{\mathcal{R}}$ really are Hahn power series fields, $\tilde{\mathcal{R}}$ is the cauchy completion of $\mathcal{R}$ and lies between the other two. As this chapter does not produce any extraterrestrials, though it does eliminate a few, this is left as a task for Chapter 7.

Chapter 8: nonstandard structures for superreal fields. The main idea appears. This reviewer (or, if you prefer, this model theorist) would have liked to see at least some of the clean discussion in Chapter 10 moved up to this point and a systematic use of the Łoś idea. As a first application, the gap theorem (8.61) comes in at this point. Then in Chapter 9, something quite different: a very nice proof that (Theorem 9.26) the particular field $\tilde{\mathcal{R}}$ is not only an extraterrestrial, but is hyperreal (though not an ultrapower of $\mathbb{R}$). Under GCH, this can be done somewhat indirectly. It suffices to build a hyperreal field K (in other words, a suitable topological space X and a suitable maximal ideal in $C(X)$) which is cauchy complete and has a dense subset of cardinality $\aleph_1$, because under CH this is then isomorphic to $\tilde{\mathcal{R}}$ (Theorem 3.21 (iv)). Both of these conditions – cauchy completeness and the density condition – translate back into the "nonstandard" structure on the hyperreal field and lead to combinatorial conditions that can eventually be met. Note that the additional nonstandard structure is used here not to differentiate structures but to identify them.

At this point an effort which has been made to keep too heavy a dose of logic out of the discussion begins to break down as unwieldy. As a result we get a rather brief and clear description of the issues at hand in Chapter 10 and some tricky technicalities concerning the coding of binary sequences, followed by what amounts to a journal article in Chapter 11 where few punches are pulled, and it is shown that the models of arithmetic and analysis we have been looking at are occasionally recoverable in all their glory from the field structure, which suggests that a purely algebraic description of the situation has indeed gone way "over the horizon".

I found some misprints, primarily of the most harmless variety. Here are all the ones one might stop for (for some I stopped long enough to ask Woodin). On p. 152, Definition 7.11, the principle "diamond" is misstated (σ_α is a set of ordinals) but used correctly in the following lemma; one might find this confusing, in which case the work of Kunen referenced in [2] is the place to go. Proposition 6.25 has gotten a bit turned around, and starting from the expression "coi" (coinitiality) in the fourth line of the six line proof, one should convert that to "cof" (cofinality) and go on in that vein. In the second paragraph on p. 164, one is referring to non-P-points. In the proof of Theorem 8.35, it is not the case that the set called J is an initial segment as stated, except in special cases, but it is certainly an interval, and the proof will go through on that basis, with some rewording at the end and in the following remark. One should also bear in mind throughout that the term "field" is taken to include "$\mathbb{R}$-algebra", and that accordingly one occasionally needs a term for an ordinary "field": the term used is "rational field".

As noted earlier, the problem concerning the possibility that every z-ideal η_1 field is hyperreal remains open (p. 342). Hmm ...

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