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Black holes - gravitational interactions, by P. D. d'Eath, Oxford Scientific Publications, Clarendon Press, Oxford, 1996, 286 + xii, $80.00, ISBN 0-19-851479-4

Black holes are, Chandrasehkar said, the most perfect macroscopic objects, depending only on our concepts of space and time. Isolated black holes in asymptotically flat spacetimes occur in only a few forms, labeled by parameters: mass, electric charge, and spin. d'Eath and I refer to black holes in 3 + 1 dimensional spacetime exclusively. Isolated black holes are the analogue of the Newtonian point mass solutions. (The "asymptotically flat" above means that infinitely far from the black hole, Newtonian dynamics of orbits can be used to measure the black hole mass.) In contrast to the well-understood single black hole case, the interaction of fields, or other objects, or other black holes with black holes is much more difficult to describe.

There are some elegant and simple studies of the horizon structure of (even interacting) black holes. The horizon is the 3-dimensional surface traced out in 4-space by those light rays which just hover between escape to infinity and collapse into the black hole. The area of a black hole is measured at one instant by taking a spacelike section through the horizon structure. Hawking was able to show by studying the structure of the generators that the area of a black hole cannot decrease (classically; Hawking also demonstrated a *quantum mechanical* evaporation of black holes). In the case of nonrotating, static, isolated black hole, the area equals $4\pi(2GM/c^2)^2$ where G is Newton's constant, c is the speed of light, and M is the mass.

Thus Hawking's area nondecrease theorem is a refinement of the idea that if you add mass to something (a black hole) it gets heavier. Qualifications include the restriction that whatever falls in obeys an energy condition: some variant of positive energy density and/or positive sum of energy density and principal stresses. A second black hole qualifies, so dropping a second black hole into an existing one will increase the mass and the area of the existing one.

Astrophysical interest in black hole mergers has heightened because of the current development of laser interferometer gravitational wave detectors. Because a black hole has the largest possible gravitational potential, one expects the gravitational wave signal from the merger of two holes to be stronger than that from any other source. How strong? Well, Hawking's theorem says:

$$M_f{}^2 \geq M_{i1}{}^2 + M_{i2}{}^2$$

where M_f is the mass of the final black hole and M_{i1}, M_{i2} are the masses of the initial black holes. The energy radiated in the merger (gravitational radiation, from uncharged black holes) is:

$$(M_{i1} + M_{i2} - M_f)c^2 \,.$$

These two formulae together give some analytic estimates of radiation efficiency.

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A recent effort (including a computational grand challenge) has attacked the problem of estimating radiation efficiency. The computational questions are precisely complementary to the global approach described in the theorems. Computational approaches require precise specification of coordinate (gauge) conditions, preparation of initial data for complex situations, and then a 3 + 1 evolution of these data. To the usual panoply of numerical difficulties, we find that the analytic detail required to understand the situation is formidable. A large computational infrastructure to support these computations has been developed, and the whole structure is just beginning to be fruitfully used.

d'Eath shows us that there is a middle ground between purely global estimates and the purely computational. He considers the fact that rapidly moving black holes have a simpler structure than do static black holes, and he therefore studies the collision of *relativistically* boosted (i.e., speed approaching the speed of light) black holes. In such a limit the black hole's observable properties become those of plane waves, and the reduction to planarity allows a wealth of known properties of colliding plane gravitational waves to be incorporated.

Then how to calculate the radiation efficiencies in black hole processes? Approaches available are:

- perturbation theory: weak fields interacting with black holes, scattering of weak gravitational waves, generation of waves in small perturbations of a black hole, e.g. by a small hole falling into a large one;
- computational approaches: attempts to describe the full nonlinear dynamics of the Einstein equation in a discretized form;
- approaches like those pioneered by d'Eath: an attempt to extend the global approaches to the case of strongly interacting black holes and to extract detail about waveforms and energy emission from such encounters.

d'Eath's approach is the inverse of usual perturbation theory. The usual theory assumes that there is a given background and something small happens to give small changes from that background. For instance, the problem of a small black hole dropping into a large black hole was carried through in 1971 by Davis, Ruffini, Press, and Price [1]. It provides a cautionary tale on the use of the Hawking theorem's upper limits: From the formulae above, one can consider the perturbation limit. If $M_{i1} \gg M_{i2}$,

$$\begin{aligned} M_f &> M_{i1}\sqrt{1 + {M_{i2}}^2/{M_{i1}}^2}\,, \\ M_f &\mathrel{\tilde{>}} M_{i1} + {M_{i2}}^2/(2M_{i1})\,, \end{aligned}$$

and the energy radiated is bounded by

$$M_{i1} + M_{i2} - M_{i1} - {M_{i2}}^2/(2M_{i1})\,.$$

While this suggests that

$$M_{i2}(1 - M_{i2}/(2M_{i1}))c^2$$

can be radiated (i.e., almost all of M_{i2}), this upper limit is a vast overestimate; the radiation is of order $M_{i2}\left(\frac{M_{i2}}{M_{i1}}\right)$, as can be shown by direct perturbation computation.

In the opposite limit (equal mass black holes) perturbation theory cannot be applied. But the Hawking limit for the interaction of equal mass black holes gives:

$M_{i1} = M_{i2} \equiv M_i/2$ and $M_f > \sqrt{2}M$. The energy radiated is then bounded by

$$(1 - \sqrt{2}/2)(M_i/2)c^2 \cong 0.29 M_i c^2.$$

A 29% efficiency is substantial in the astrophysical context. Axi-symmetric computations (axi-symmetric means "head-on collision") in the '70s by Smarr [2], and repeated several times since, give roughly a factor of 1,000 less radiation than this estimate. A very important astrophysical question thus concerns the non-head-on limit.

d'Eath's program is to consider an expansion away from $v = c$. The lowest-order case consists of the collision of two plane waves, the behavior of which is well understood. By this method he computes the forward gravitational radiation from ultra-high-speed encounters, which would give a radiation efficiency of 25% *if* it were part of an isotropic pattern. (There is some theoretical justification for making this assumption; see Smarr [3]). Thus d'Eath is able to give the dominant early features of the gravitational radiation produced in such encounters. By considering smaller features that arise because the black holes are not moving at exactly the speed of light, d'Eath attempts to include more and more detail in the interaction. However, those results are approximate; the next order of the expansion by this technique is not in a well-understood state, and d'Eath's approximation gives unreliable results. But the leading terms *are* understood and the advantage of d'Eath's approach is that they are *explained* in the process of carrying out the technique, a feature that is not automatic in the computational approach. Comparison to d'Eath's predictions will be an early priority of the computational work.

This book is a summary of work done by d'Eath in a number of papers, with a good survey of work also by other authors, particularly on perturbation theory. It is heavy going in places in terms of the necessary complication of perturbation theory, but it is a monograph that will be of definite interest and use to any researcher trying to understand the intricacies of gravitation wave physics.

References

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Richard A. Matzner
The University of Texas at Austin
E-mail address: richard@ricci.ph.utexas.edu