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Lévy processes, by Jean Bertoin, Cambridge Univ. Press, Melbourne, NY, 1996, x+265 pp., \$54.95, ISBN 0-521-56243-0

Let $\mathcal{P}$ be the set of probability measures on the line. An element $P \in \mathcal{P}$ is determined uniquely by its *distribution function* $F_P(x) = P((-\infty, x])$, $x \in \mathbb{R}$, and by its *characteristic function* (Fourier transform)

$$\varphi_P(\lambda) = \int e^{i\lambda x} F_P(dx), \quad \lambda \in \mathbb{R}.$$

The convolution $P * Q$ of two probability measures has distribution function

$$F_{P*Q}(x) = \int F_P(x-y)F_Q(dy) = \int F_Q(x-y)F_P(dy);$$

convolution of probability measures corresponds to multiplication of their characteristic functions. If X and Y are independent random variables, then the distribution of their sum is given by the convolution of the distributions of X and Y.

The study of the structure of the semigroup $(\mathcal{P}, *)$ is sometimes called the arithmetic of probability distributions. A probability distribution P is *indecomposable* if it has no non-trivial factors. (It is easy to find examples – for example any distribution which assigns mass to exactly two points). At the other extreme P is *infinitely divisible* (ID) if, for each $n \geq 1$, there exists $Q_n \in \mathcal{P}$ such that P is the n-fold convolution of Q_n. So P is ID if and only if for every $n \geq 1$ there exists a characteristic function φ_n such that $\varphi_P(t) = \varphi_n(t)^n$. While the theory of indecomposable distributions has made few connections with other areas of probability, that of ID distributions has been extremely fruitful.

Specific examples (such as the Gaussian, Cauchy and Poisson) have been known for a long time, but the systematic study of ID distributions began in the 1920s and 1930s, with the work of Lévy and Khintchine. From almost the beginning the connection with stochastic processes was apparent. Let P_1 be an infinitely divisible law, and let X_1 be a random variable (on a suitable abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$) with distribution P_1. Since P_1 is the n-fold convolution of a law P_n, we have (modifying the probability space if necessary) that there exist independent random variables Y_i, $1 \leq i \leq n$, such that $X_1 = \sum_{i=1}^n Y_i$. Thus X_1 is the value at time 1 of a stochastic process $(X_t, t \in \{0, 1/n, 2/n, \ldots, 1\})$, defined by

$$X_{k/n} = \sum_{i=1}^{k} Y_i, \quad 1 \leq k \leq n. \tag{1}$$

It is easy to see that X has *stationary independent increments*. That is, the law of $X_t - X_s$ depends only on $t - s$, and if $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k$, then

$$X_{t_i} - X_{t_{i-1}}, \quad 1 \leq i \leq k, \text{ are independent random variables.}$$

Naturally, one wants to let $n \to \infty$ in the above argument, to obtain a process X with stationary independent increments and parameter set $[0,1]$. This can be done, using Kolmogorov's extension theorem. (Some additional work is necessary to

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obtain a 'good version' of X, for which the sample paths $t \to X_t(\omega)$ are measurable.) Extending the time set to $[0,\infty)$ is easy, and we deduce that each ID distribution corresponds to a *process with stationary independent increments*, or, for short (since this is rather a mouthful), a *Lévy process*.

The analytic characterization of ID distributions was obtained by Lévy and Khintchine (see [L1]); the *Lévy-Khintchine formula* states that if P is ID, then $\varphi_P(\lambda) = e^{\psi(\lambda)}$ where

$$\psi(\lambda) = ia\lambda - \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{\infty} \big(e^{i\lambda x} - 1 - i\lambda x I_{(|x|<1)}\big)\pi(dx), \quad \lambda \in \mathbb{R}. \tag{2}$$

Here a and $\sigma \geq 0$ are real, and π (the *Lévy measure*) is a measure on $\mathbb{R} - \{0\}$ such that

$$\pi(\mathbb{R} - [-1,1]) < \infty, \text{ and } \int_{-1}^{1} x^2 \pi(dx) < \infty.$$

(The function $xI_{(|x|<1)}$ can be replaced by other bounded functions with equivalent behaviour at 0 – another popular choice is $x/(1+x^2)$. Changing this function changes the constant a.)

Taking $a = \pi = 0$, $\sigma = 1$ in (2) gives a standard Gaussian distribution, while $a = \sigma = 0$, $\pi = p\delta_1$ (where δ_1 is the unit point mass at 1) yields the Poisson distribution with parameter p. The corresponding stochastic processes are Brownian motion and the Poisson process. The first, as is well known, has continuous non-differentiable sample paths, while the Poisson process is an increasing process which makes jumps of size 1 at independent exponential times.

If ν is a probability distribution on $\mathbb{R} - \{0\}$, then a *compound Poisson process* is obtained by replacing the jumps of size 1 by independent jumps with distribution ν: we obtain a characteristic function with exponent

$$\psi(\lambda) = \int_{-\infty}^{\infty} \big(e^{i\lambda x} - 1\big) p\nu(dx). \tag{3}$$

These examples suggest that moving from the level of random variables and characteristic functions to that of stochastic processes will give an intuitive meaning to the terms in the Lévy-Khintchine formula (2), and this is indeed the case, modulo one difficulty. The sum of independent Lévy processes is a Lévy process, so we can combine different 'building blocks' to obtain a general characteristic function of the form (2): a represents a deterministic drift, σ^2 a Gaussian process of the form σW_t where W is a standard Brownian motion, and π a jump measure. Everything is clear and intuitive, except for the term $i\lambda x I_{(|x|<1)}$: this arises because to obtain a convergent sequence of compound Poisson processes with Lévy measures $p_n\nu_n \to \pi$, it may be necessary to 'compensate the jumps' by subtracting drifts $a_n = \int_{(-1,1)} x p_n \nu_n(dx)$.

Thus there is an exact correspondence between:
(ID) Infinitely divisible laws on $\mathbb{R}$,
(LK) Probability measures with characteristic function given by (2),
(LP) Lévy processes.

Two of these implications ((LK)$\Rightarrow$(ID) and (LP)$\Rightarrow$(ID)) are trivial; one could build a graduate probability course around the other four, each of which takes one through different territory. (ID)$\Rightarrow$(LP) involves the Kolmogorov extension theorem, regularisation of paths and fundamentals of Markov process theory. The direct proof of (ID)$\Rightarrow$(LK) uses the analytic theory of characteristic functions – see for example [F, Chapter XVII] – while (LK)$\Rightarrow$(LP) proceeds as outlined above. Finally (LP)$\Rightarrow$(LK) starts with a Lévy process and 'pulls off' the jumps, which have to be compound Poisson, until a continuous Lévy process is left, and this (using the central limit theorem) must be Gaussian. The last two approaches are those used by Lévy and Itô in [L1], [L2], [I]. While probabilistically intuitive, they do involve some machinery which takes a while to develop – Poisson point processes and martingale convergence. (Of course, the machinery is worth developing anyway....)

Two classes of Lévy processes deserve special mention. A *strictly stable process of index* $\alpha \in (0, 2]$ is a process such that X_t and $\lambda^{-1/\alpha} X_{\lambda t}$ are equal in distribution. These have Lévy measure given by

$$\pi(dx) = (c_+ I_{(x>0)} + c_- I_{(x<0)}) x^{-\alpha-1} dx, \quad \alpha \neq 1, 2. \tag{4}$$

(The literature on stable distributions contains many minor errors in the constants, particularly in the case $\alpha = 1$ – see [H].) The tail of the measure π in (4) is fat, and this means that when $\alpha < 2$, stable processes have infinite variance, due to the possibility of very large jumps. These processes have the interesting property that the value of the process at time t is comparable with the size of the largest jump in $[0, t]$.

A *subordinator* is a Lévy process with increasing sample paths. These arise quite frequently in theoretical applications: for example the set of times at which a standard Brownian motion returns to 0 is the range of a stable subordinator of index 1/2.

The theory of Lévy processes serves as an excellent introduction to that of Markov processes, giving much of the flavour of the subject, but without the heavy technical preliminaries of the general theory. In addition, not only are Lévy processes important in their own right, but they can also arise as components of more complicated systems, such as branching processes and random trees – see for example [B], [LL].

Lévy processes are also natural models for phenomena with discontinuities. To mention just one example, Mandelbrot (see [M1, p. 337]) suggested that stock prices could be modelled by stable processes with index $\alpha < 2$. Certainly a model allowing jumps (which would occur on the arrival of new information) is plausible on *a priori* grounds and is to some extent confirmed by observation. (With more large downward jumps than upward?)

However, the majority of financial models, used for example in hedging options, use (logarithmic) Brownian motion: it may be the wrong model, but it is much easier to calculate with. A simple example will suffice. If $a, b > 0$ and I is the interval $[-a, b]$, then a Brownian motion started at 0 can leave I in two ways (upward or downward), and the probability of each is easily found. But a general Lévy process can leave I in four ways (up or down, continuously or by jumping), and in general these probabilities have no simple form. More complicated barrier problems for Brownian motion reduce to solving a 2nd order PDE, while the corresponding problem for a Lévy process requires an integro-differential equation.

Discontinuous models have one other important consequence – perfect hedging is impossible. (The technical term is an 'incomplete market'.) If the stable model were correct, then portfolio insurance would be a disaster.

There is a general dearth of books on specific classes of Markov process, and this volume, the first comprehensive treatment of Lévy processes in English, fills a gap which has been evident for some time. ([S] is in Japanese, though an English translation is in preparation.) The book consists of essentially two halves. The first five chapters deal with basic theory. After two chapters on preliminaries, including a quick and neat construction of Lévy processes from Poisson point processes, Chapter II covers potential theory. Chapters III and IV discuss subordinators (increasing Lévy processes) and their application to the local times of Markov processes.

The final four chapters discuss some more specialized topics. Chapter V looks at the global behaviour of the local times – this includes an account of Kesten's criterion for regularity of points. Chapter VI looks at 'fluctuation theory', that is, the description of the joint distribution of X_t and $\sup_{s\leq t} X_s$. The formulae in Chapter VI take on a much more tractable form if the Lévy measure is *spectrally negative* – i.e. $\pi((0,\infty)) = 0$. These processes are the subject of Chapter VII. Finally Chapter VIII deals with (strictly) stable processes, their sample path properties, and the definition and properties of stable bridges. Much of this material is due to the author or his co-workers and has appeared in the last decade.

Overall this book provides an excellent introduction to the subject and could be used as a textbook for an advanced graduate course. Though reasonably self-contained, it does presuppose a degree of probabilisitic sophistication in the reader in its use of tools such as excursion theory.

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