BOOK REVIEWS

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The book of involutions, by M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol (with a preface by J. Tits), American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998, xxi + 593 pp., $69.00, ISBN 0-8218-0904-0

The title of this book might seem a bit mysterious or ambiguous, the word 'involution' having several connotations. A glance at the first line of the Introduction dispels the mystery: the reader learns there that involutions on central simple algebras and symmetric or alternating bilinear forms are central themes of the book. The title *The algebra of classical groups* would also have been quite appropriate, as classical groups are visible or in the background throughout the book. (But I like the present title.)

The first occurrence of the name 'classical groups' was apparently in 1939, when H. Weyl published his book [Wey] with the same title. The groups of that book are general (or special) linear groups, and orthogonal and symplectic groups, in the first instance over the field of real numbers. Over finite fields such groups had already been introduced and studied by C. Jordan in the 19^{th} century and by L. E. Dickson around 1900 (see [Dic]). Subsequently, classical groups over arbitrary commutative fields were taken up by J. Dieudonné around 1948 (see [Die]). In order to deal with, say, orthogonal groups over an arbitrary field, one needs general results from the theory of quadratic forms (like Witt's theorem). As a natural outgrowth, such results were generalized to classical groups over non-commutative fields. There is a close connection between (generalized) classical groups over central simple division algebras and finite dimensional semisimple algebras with involution, as was pointed out by A. Weil in 1960 in [Wei] (he ascribed this observation to Siegel; see [loc. cit., p. 549]). It follows that the generalized classical groups can be viewed as groups of rational points over a ground field, of suitable quasi-simple algebraic groups of the 'classical' types A, B, C, D over that field. Thus the theory of classical groups was put in the context of the theory of linear algebraic groups, which had been developed in the sixties by Borel and Chevalley. Also, the 'exceptional' algebraic groups of types G_2, F_4 and those of 'trialitarian' type D_4 turned out to have close connections with quadratic forms and simple algebras of low dimension.

In the meantime, a great deal of work has been done and many things have been clarified. The present book gives an up to date and quite complete exposition

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of the aspects of algebra theory related to classical groups, including the matters mentioned in the previous paragraph.

In the first chapter, 'Involutions and Hermitian Forms', the stage is set. Let A be a central simple algebra over a field F. This means that A is an algebra over F such that there exists an extension K of F with $K \otimes_F A \simeq \mathbf{M}_n(K)$, a full matrix algebra over K. The integer n is the degree of A.

An *involution* of A is an anti-automorphism σ with $\sigma^2 = 1$. It is of *the first kind* if the restriction to the center F is the identity. Otherwise σ is of the second kind or *unitary*. Involutions of the first kind (at least if the characteristic of F is $\neq 2$) are the incarnations in algebra theory of quadratic forms and symplectic forms (and their generalizations). More precisely, if $\mathrm{char}(F) \neq 2$ and σ is of the first kind, there is an extension K of F such that $K \otimes_F A$ is isomorphic to $B = \mathbf{M}_n(K)$ and that σ induces on B an involution $x \mapsto b({}^t x)b^{-1}$, where b is a non-singular symmetric or skew-symmetric matrix. σ is then called orthogonal or symplectic. In characteristic 2 symplectic involutions of the first kind are defined in the same manner. The unitary involutions are incarnations of Hermitian forms.

This chapter discusses basic facts, such as the characterization of algebras which admit an involution of the first or second kind, in terms of the Brauer group. Knowledge is assumed of basic facts about central simple algebras. These are recapitulated in the beginning of the chapter.

The set-up of algebras with involution does not englobe quadratic forms in characteristic 2. To deal with these, the authors, inspired by ideas of J. Tits (see [T]), introduce a (new) notion of quadratic pair. Let σ be an involution of the first kind on A, as before. The authors give a characteristic free definition of the notion of a quadratic pair (σ, f) on A. Let $S = \{a \in A \mid \sigma(a) = a\}$, the set of σ-symmetric elements. In characteristic $\neq 2$ giving a quadratic pair (σ, f) is the same thing as giving an orthogonal involution of the first kind σ together with an invertible element of S, up to a non-zero scalar. If $A = \mathbf{M}_n(F)$, this is the same as giving a non-singular quadratic form, up to a scalar.

If $\mathrm{char}(F) = 2$, (σ, f) is a quadratic pair on A if

(a) σ is a symplectic involution of A,

(b) f is a linear function on S such that $f(x + \sigma(x)) = t(x)$ $(x \in A)$, where t is the reduced trace on A.

If $A = \mathbf{M}_n(F)$, giving a quadratic pair again amounts to giving a non-singular quadratic form, up to a non-zero scalar.

The second chapter, 'Invariants of Involutions', introduces invariants of involutions and quadratic pairs, generalizing classical invariants of quadratic forms.

The first invariant is the index of a central simple algebra with involution or of a quadratic pair over F (as before). This is the analogue of the Witt index of a quadratic form.

Next comes the discriminant of an orthogonal involution of the first kind in characteristic $\neq 2$. This lies in $F^*/(F^*)^2$. A quadratic pair in characteristic 2 also has a discriminant, lying in $F/\wp F$, where $\wp x = x + x^2$ $(x \in F)$.

After recalling the definition and properties of the Clifford algebra of a quadratic form, the Clifford algebra $C(A, \sigma, f)$ of a quadratric pair (σ, f) on A is introduced and studied. This algebra generalizes the *even* Clifford algebra of a quadratric form. The analogue of the odd part of such an algebra is a bimodule over $C(A, \sigma, f)$.

In the case of a central simple algebra (A, σ) with a unitary involution whose center is a separable quadratic extension of its field F of σ-fixed points, the study of discriminants leads to a notion of *discriminant algebra* $D(A,\sigma)$, which is an algebra over F with an involution of the first kind. In the case that A is the full matrix algebra, $D(A,\sigma)$ is a quaternion algebra over F.

The matters discussed in this chapter, some of them quite subtle, are clearly explained.

The first two chapters, comprising about a quarter of the book, are essentially about simple algebras. In the next chapter, 'Similitudes', the (generalized) classical groups make their appearance.

Let (A,σ) be a central simple algebra with involution. Denote by K the center and by F the subfield of K whose elements are fixed by σ. The group of isometries $\mathrm{Iso}(A,\sigma)$ is the group of invertible $x \in A$ with $\sigma(x) = x^{-1}$. The group of invertible $x \in A$ with $\sigma(x)x \in F$ is the group of similitudes $\mathrm{Sim}(A,\sigma)$. If $A = \mathbf{M}_n(K)$ and σ is of the first kind and orthogonal in characteristic $\neq 2$ (respectively: of the first kind and symplectic, unitary), then $\mathrm{Iso}(A,\sigma)$ is an orthogonal group (respectively: a symplectic group, a unitary group) and $\mathrm{Sim}(A,\sigma)$ is the corresponding group of similitudes.

If (σ, f) is a quadratic pair on A in characteristic 2, the orthogonal group $O(A,\sigma,f)$ is the group of $x \in \mathrm{Iso}(A,\sigma)$ with $f(xsx^{-1}) = f(s)$ for all s fixed by σ. The group of orthogonal similitudes $GO(A,\sigma,f)$ is defined similarly.

The main item of this chapter is the study of the action of orthogonal similitudes on the corresponding Clifford algebras, leading to spin groups and similar groups.

For orthogonal groups (in characteristic $\neq 2$) of dimension $3,4,5,6$ the Clifford algebra admits another description, which leads to a connection of such a group with a different classical group, a remark which seems to go back to Eichler (in [E, §5]). These matters are discussed, in the general framework adopted in the book, in Chapter IV ('Algebras of Degree Four'). The results are formulated in terms of equivalences of groupoids (categories in which all morphisms are isomorphisms). The notations anticipate the description of classical groups by the type of their Dynkin diagram.

Here is a sample result. For $n \geq 2$ let $\mathbf{A}_n$ be the category of quaternion algebras with a unitary involution, over an étale quadratic extension of the ground field F, the morphisms being F-isomorphisms preserving involutions. (An étale quadratic extension of F is either a separable quadratic extension, or is isomorphic to $F \oplus F$. The notion of a central simple algebra with a unitary involution over $F \oplus F$ has an obvious definition.) Denote by $\mathbf{D}_n$ the category of central simple F-algebras of degree $2n$ with a quadratic pair, where the morphisms are F-algebra isomorphisms preserving quadratic pairs. Then the categories $\mathbf{D}_3$ and $\mathbf{A}_3$ are equivalent. The proof uses the Clifford algebra construction on the second category and the discriminant algebra construction (of the second chapter) on the first category. As an application of this equivalence of categories a proof is given of Albert's theorem that a central simple algebra of degree 4 and exponent 2 is a biquaternion algebra (i.e. a tensor product of two quaternion algebras).

The second part of the chapter is devoted to a finer study of biquaternion algebras and their Whitehead groups.

The next chapter, 'Algebras of Degree Three', discusses another low dimensional situation. The chapter starts with a discussion of unital commutative étale algebras over a field, leading to the equivalence between the category of these algebras and the category of finite sets with a continuous action of the profinite Galois group

$\mathrm{Gal}(F_s/F)$, where F_s is a separable closure of F. The case of three dimensional algebras is discussed in detail.

The main topic of the chapter is the discussion of central simple algebras of degree 3 with a unitary involution. First Wedderburn's theorem is proved: a central simple algebra of degree 3 is cyclic. Then it is shown that the conjugacy classes of involutions on a central simple algebra A over an étale quadratic extension K of F, which fix the elements of F, are classified up to conjugacy by an 8-dimensional quadratic form over F (a 3-fold Pfister form). The involution is distinguished if this quadratic form is hyperbolic. It is shown that if (A, τ) is as above, there always exists a distinguished involution τ' on A.

Next the algebraic groups make their appearance. The chapter 'Algebraic Groups' first discusses basic facts about linear algebraic groups over a field F, or rather about affine group schemes over F. The approach is via Hopf algebras. Let A be a commutative Hopf algebra over F, with a counit and an antipode. Assume it to be finitely generated over F. The Hopf algebra axioms express the facts that for any commutative F-algebra R the set $G^A(R)$ of F-homomorphisms $A \to R$ is a group, and that G^A defines a functor from the category of commutative F-algebras to the category of groups. G^A (and any functor G of the same kind isomorphic to some G^A) is an *affine group scheme* over F. G^A is an *algebraic group* over F if for any extension E of F the algebra $E \otimes_F A$ is reduced (i.e. is without nilpotent elements $\neq 0$). The first part of the chapter gives a brisk review of the basic facts about group schemes, many with proofs. The group schemes and algebraic groups associated to algebras with involution are discussed.

Absent are the algebro-geometric aspects of the theory of algebraic groups involving non-affine varieties, like Borel's fixed point theorem, flag manifolds. (Curiously, projective algebraic geometry makes a fleeting appearance in Chapter I, where Grassmannians and Severi-Brauer varieties are introduced, rather tersely. However, it is also pointed out that these objects are not really needed there.)

The second part of the chapter is devoted to the classification of semisimple groups. First basic facts about root systems and their classification are reviewed. Next comes the classification of split semisimple groups (over F), in terms of their root systems. Reductive groups are not mentioned, which is a bit surprising as there are many examples of these in the context of classical groups.

The semisimple groups with an irreducible root system of one of the types $A_n,\ B_n,\ C_n,\ D_n\ (n \neq 4),\ F_4,\ G_2$ are identified in terms of classical groups (respectively, exceptional Jordan algebras and Cayley algebras). The identification is again in terms of equivalences of categories. Here is an example: let $\mathbf{D}^n$ be the groupoid of simply connected, quasi-simple groups over F of type $D_n, n > 4$, the morphisms being F-isomorphisms. This category is equivalent with the category $\mathbf{D}_n$ introduced above.

The chapter ends with a discussion of Tits algebras. These are central simple algebras over F associated to (suitable) irreducible representations of G. It is shown that the discriminant algebra of a central simple algebra with a unitary involution can be viewed as a Tits algebra. Likewise the Clifford algebra of a quadratic pair.

This chapter reviews a great deal of technical material about algebraic groups. Although well-written, it will perhaps be heavy going for readers who are not well versed in the arcana of algebraic groups. (For example, on p. 364 the reader is assumed to know that a semisimple group over F contains a maximal torus over F.)

The next chapter, 'Galois Cohomology', discusses the identification of isomorphism classes of algebraic objects (central simple algebras, algebras with involution...) in terms of Galois cohomology. For a group scheme G over F the Galois cohomology set $H^1(F,G)$ is introduced. Basic results about the formalism of Galois cohomology are established.

The chapter ends with a review (without full details) of recent results about cohomological invariants of an algebraic group G over F. These are maps (with good functorial properties) $H^1(F,G) \to H^d(F,M)$, where M is a suitable Galois module. (A Galois module for F is an abelian group with a continuous action of the Galois group Γ of a separable closure F_s/F, relative to the discrete topology on M. Then $H^d(F,M)$ is the cohomology group $H^d(\Gamma,M)$.) Cohomological invariants provide a connection between non-commutative Galois cohomology of G and commutative Galois cohomology. Assume G to be absolutely simple and simply connected. There is a very interesting invariant with $d=3$ discovered by Rost, with $M=\mu_N \otimes \mu_N$, where μ_N is the group of N^{th} roots of unity, N depending on G. Its properties are discussed, and it is identified in the case of classical groups.

From the perspective of algebraic groups the last three chapters—'Compositions and Triality', 'Cubic Jordan Algebras' and 'Trialitarian Central Simple Algebras'—do not deal with classical groups, but with 'exceptional' groups, namely the simple groups of respective types G_2, F_4 and of the trialitarian type D_4. The chapters give a confirmation of Weil's hope (see [Wei, p. 549]) that these groups also have something to do with algebras with involution. The groups are related to automorphism groups of certain exceptional algebraic structures over a field F. These are, respectively, Cayley algebras, exceptional simple Jordan algebras and trialitarian algebras (a new notion). In the analysis of these structures classical algebraic structures play an important role, viz. threefold Pfister quadratic forms in the case of Cayley algebras and central simple algebras of degree three, possibly with a unitary involution, in the other cases. These chapters bring a very welcome addition to the literature. They contain the first coherent exposition in a textbook of the exceptional algebraic structures of the types mentioned before.

All chapters contain exercises (of varying degrees of difficulty) and detailed historical notes. There is an extensive bibliography.

The book is well-written. Although it has four authors, the style is surprisingly uniform. There are some differences in style in the chapters, though, probably caused by the differences in character of the chapters. Thus, the first two chapters contain a complete and reasonably elementary treatment of the algebraic theory of central simple algebras with involutions. Likewise, the last three chapters give a treatment of exceptional algebras, largely independent of the immediately preceding chapters. The exposition in the last chapters is rather more leisurely than in the chapter 'Algebraic Groups', which contains a condensed review of an elaborate theory. (It would perhaps have been useful if the authors had added a 'Leitfaden', indicating the interconnections of the chapters.)

My overall opinion is that this is a very valuable book. I expect that it will be the standard text on the algebra of classical groups in the near future.

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