BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 3, Pages 373–377
S 0273-0979(00)00872-7
Article electronically published on April 7, 2000

Fourier analysis on number fields, by D. Ramakrishnan and R. J. Valenza, Springer, New York, 1999, xxi + 350 pp., \$39.95, ISBN 0-387-98436-4

Dinakar Ramakrishnan and Robert J. Valenza's recent book *Fourier Analysis on Number Fields* ([RV]) is an introduction to number theory organized around John Tate's 1950 Princeton Ph.D. thesis [T]. Less comprehensive than Weil's famous book *Basic Number Theory* ([W]), Ramakrishnan and Valenza's book is notable for the thoroughness with which it treats the analytic background necessary to fully appreciate the technicalities of Tate's methods.

Tate's thesis, "Fourier Analysis in Number Fields and Hecke's Zeta-Functions", combined techniques of abstract Fourier analysis with the valuation-theoretic approach to algebraic number theory developed by Chevalley and Artin and Whaples to obtain an extremely powerful and elegant approach to the theory of zeta functions. The approach pioneered by Tate in his thesis has proved to be extremely influential in many later developments in number theory. Ramakrishnan and Valenza convey an appreciation for the depth of the analysis underlying Tate's work and the power and clarity of Tate's methods. The reader who masters the material in Ramakrishnan and Valenza's text will have no trouble going on to read the more general results, including the proofs of the main theorems of Class Field Theory, as given in Weil's book.

The zeta functions studied in Tate's thesis are Hecke's generalization of the zeta and L-functions of Riemann, Dedekind, and Dirichlet. Let K be a number field, meaning a finite extension of the rational numbers $\mathbf{Q}$, and let $\mathcal{O}_K$ be the ring of algebraic integers in K. Let d denote the degree of K over $\mathbf{Q}$. The fractional $\mathcal{O}_K$-ideals form a group I isomorphic to the free abelian group on the non-zero prime ideals of $\mathcal{O}_K$, and the principal ideals (those generated by a single element of K) form a subgroup of this group. To define a Hecke character, fix an integral ideal M of K and let $I(M)$ be the fractional ideals prime to M. A Hecke character is then a homomorphism

$$\Psi : I(M) \to \mathbf{C}^*$$

that satisfies the following condition: there exist integers $m_1, \dots , m_d$ and complex numbers $s_1, \dots , s_d$ such that if $\alpha \in K^*$ is congruent to one modulo M, (α) denotes the principal ideal generated by α, and $\sigma_1, \dots , \sigma_d$ are the embeddings of K into $\mathbf{C}$, then

$$\Psi((\alpha)) = \prod_{j=1}^{d} \sigma_j(\alpha)^{m_j} |\sigma_j(\alpha)|^{s_j}.$$

If $|\Psi| = 1$, then Ψ is said to be unitary. The Hecke L-function associated to Ψ is then the sum

$$L(s, \Psi) = \sum_{\substack{I \subseteq \mathcal{O}_K \\ (I,M)=1}} \frac{\Psi(I)}{N(I)^s}$$

2000 *Mathematics Subject Classification.* Primary 11-01, 11R42, 11F30, 11R37, 20E08, 20E18.

where the sum is over non-zero integral ideals of $\mathcal{O}_K$ prime to M, $N(I)$ is the norm of the ideal I, and s is a complex variable. This series converges in a suitable half-plane in s.

The prototypical example of a Hecke L-function is the Riemann zeta function, obtained by choosing $K = \mathbf{Q}$, M the unit ideal, and Ψ trivial. Slightly more generally, if one chooses all of the parameters m_j and s_j equal to zero in the definition of Ψ, one obtains a character of finite order. An example of such a Ψ for the field $\mathbf{Q}$ may be constructed by choosing an integer m and a multiplicative homomorphism

$$\Psi : (\mathbf{Z}/m\mathbf{Z})^* \to \mathbf{C}^*$$

that induces in an obvious way a homomorphism from the fractional ideals of $\mathbf{Q}$ prime to m to $\mathbf{C}^*$. The L-functions associated to characters of finite order were introduced by Dirichlet in his study of the analytic class number formula and equidistribution of primes among arithmetic progressions. More general Hecke characters are central to the theory of complex multiplication of elliptic curves and abelian varieties.

Hecke was able to show that the L-series $L(s, \Psi)$ has a meromorphic continuation to the entire complex plane and, when Ψ is unitary, $L(s, \Psi)$ satisfies a functional equation of the form

$$L(s, \Psi^*) = (\text{complicated factors})L(1 - s, \Psi)$$

where Ψ^* is another unitary Hecke character associated to Ψ. This vastly generalizes Riemann's functional equation $\xi(s) = \xi(1 - s)$ for the function

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s).$$

Hecke's proof relies on an intricate theory of theta functions.

Tate's contribution to the theory of Hecke L-functions was to reinterpret Hecke's results in terms of abstract Fourier analysis on the group $\mathbf{A}_K$ of adeles of K. The group $\mathbf{A}_K$ is the restricted direct product of all completions of the field K. In other words, an element a of $\mathbf{A}_K$ is a vector $(a_v \in K_v)$ indexed by the valuations of K with the property that $|a_v|_v \leq 1$ for all but finitely many v. The adeles are a group under componentwise addition. Also important in Tate's work is the closely related multiplicative group $\mathbf{I}_K$ of ideles, consisting of vectors $(a_v \in K_v^*)$ such that $|a_v|_v = 1$ for all but finitely many v. Both $\mathbf{A}_K$ and $\mathbf{I}_K$ are equipped with topologies making them abelian locally compact topological groups. The additive and multiplicative groups of the field K embed diagonally in $\mathbf{A}_K$ and $\mathbf{I}_K$ respectively.

In very brief outline, Tate proceeds by reinterpreting Hecke characters as continuous homomorphisms

$$\chi : \mathbf{I}_K/K^* \to \mathbf{C}^*.$$

Such homomorphisms, called quasicharacters by Tate, encode both the traditional Hecke character and the variable s. He then introduces an appropriate class of functions on the adele group $\mathbf{A}_K$ and for f in this class defines a zeta-integral as a function of the quasicharacter χ

$$Z(f, \chi) = \int_{\mathbf{I_K}} f(x)\chi(x)d^*x.$$

Here d^*x is a carefully normalized Haar measure on $\mathbf{I}_K$, viewed as a subset of $\mathbf{A}_K$, and the zeta integral $Z(f, \chi)$ is defined in a "half-plane" of quasicharacters. Tate then applies an abstract version of Poisson summation on $\mathbf{A}_K$ relative to the discrete subgroup $K \subset \mathbf{A}_K$ to obtain a meromorphic continuation of $Z(f, \chi)$ and a simple functional equation stating that $Z(f, \chi) = Z(\hat{f}, \chi^*)$. Here $\hat{f}$ is the Fourier transform of f relative to a Haar measure on $\mathbf{A}_K$ and χ^* is a related quasicharacter. In the process Tate determines the poles and residues of $Z(f, \chi)$, expressing them in terms of volumes of certain subsets of $\mathbf{I}_K^*/K^*$; explicit computation of these volumes yields the analytic class number formula.

Tate combines these computations with a parallel set of local computations carried out at each completion K_v of K. He defines local zeta-integrals $Z_v(f_v, \chi_v)$ for local functions f_v and characters χ_v at each K_v and obtains functional equations for these local functions.

Finally, he chooses the functions f and f_v cleverly and compares the local and global computations. In this way he relates his computations to Hecke's zeta-functions. He also accounts for the complicated factors arising in Hecke's functional equation as a relatively simple product of local factors derived from the local functional equations.

Tate ([T, 1.3]) describes the mathematical prerequisites for his thesis as follows:

> In number theory we assume only the knowledge of the classical algebraic number theory and its relation to the local theory.... Concerning analysis, we assume only the most elementary facts and definitions in the theory of analytic functions of a complex variable. No knowledge whatsoever of classical analytic number theory is required. Instead, the reader must know the basic facts of abstract Fourier analysis in a locally compact abelian group G:
>
> (1) The existence and uniqueness of a Haar measure on such a group, and its equivalence with a positive invariant functional on the space $L(G)$ of continuous functions on G which vanish outside a compact.
>
> (2) The duality between G and its character group $\hat{G}$, and between subgroups of G and factor groups of $\hat{G}$.
>
> (3) The definition of the Fourier transform, $\hat{f}$, of a function $f \in L_1(G)$, together with the fact that, if we choose in $\hat{G}$ the measure which is dual to the measure in G, the Fourier Inversion Formula holds (in the naive sense) for all functions for which it could be expected to hold....

Essentially the same prerequisites are required for Weil's *Basic Number Theory*; Weil assumes the basic facts about Haar measure and only sketches the main results on duality (2) and the Fourier transform (3).

The first three chapters of Ramakrishnan and Valenza's book treat Tate's points (1), (2), and (3). The first chapter is an introduction to topological groups from first principles. The highlight of this chapter is a proof of the existence of (left or right) Haar measure for a locally compact group. Chapter 2 of the book discusses the spectral theory of commutative Banach algebras. Here the goal is to develop the machinery for the proof of Schur's Lemma for topologically irreducible representations of abelian locally compact groups in Hilbert spaces. Chapter 3 applies the results of the previous two chapters to develop the theory of Pontryagin duality and the abstract Fourier transform.

These chapters of the book are accessible to a student who has had a strong first-year course in analysis. Indeed, taken together, these three chapters provide a beautiful and complete development of an important theory flowing from such standard results as the Riesz representation theorem, Alaogolu's theorem, and the Krein-Milman theorem. Chapter 3, on Pontryagin duality, is probably the deepest chapter of the text. Although the statement of the duality theorem is simple and elegant, the proof is quite sophisticated. It requires the spectral theory developed in Chapter 2, as well as fairly substantial use of convexity principles.

There is no doubt that many mathematicians, especially those whose interests tend toward the more algebraic aspects of number theory, have contented themselves with reviewing the statements of the theorems on Haar measure and the Fourier transform, and have then profitably read Tate's thesis or Weil's book. Someone who has pursued this course will find the first three chapters of Ramakrishnan and Valenza particularly educational.

Chapters 4 and 5 contain the number theoretic prerequisites to Tate's thesis, and Chapter 6 is a survey of class field theory provided mainly for motivational purposes. Chapter 4 treats the classification of locally compact fields. The existence of a Haar measure on such a field allows one to construct a valuation on the field. One then shows that, in characteristic zero, a field with valuation is either $\mathbf{R}$, $\mathbf{C}$, or a finite extension of $\mathbf{Q}_p$. The authors also show that locally compact fields of characteristic p are necessarily power series fields over finite fields. This chapter also discusses the theory of number fields and function fields. All told, Chapter 4 rather ambitiously devotes 50 pages, including exercises, to cover much of the material that would typically be treated in a semester course on algebraic number theory. Chapter 5 completes the number theoretic background by introducing the adeles and ideles, the approximation theorem and the product formula, and the various types of class group. The presentation in Chapters 4 and 5 is very similar to that of Weil.

Chapter 7 is a close reading of Tate's thesis. The reader who has mastered Chapters 1 through 5 is fully prepared to read Tate's thesis directly and will find it highly instructive to read Tate alongside Ramakrishnan and Valenza. Tate's argument maximizes efficiency, while Ramakrishnan and Valenza give results for function fields as well as number fields, and are able to replace some of Tate's *ad hoc* constructions with techniques that have become standard since Tate's thesis was written. In addition, Ramakrishnan and Valenza organize Tate's proof slightly differently and add a number of helpful clarifying remarks. Probably the most interesting example of the extra information supplied by Ramakrishnan and Valenza is their explanation of a crucial generalization of Poisson summation used in Tate's proof of the global functional equation that Tate calls "the Riemann-Roch theorem". Ramakrishnan and Valenza take the time to explain in detail the precise, non-obvious relationship between this result and the classical Riemann-Roch theorem for function fields. Finally, Ramakrishnan and Valenza give some applications of L-functions, including some results on density of primes in arithmetic progressions.

Every chapter of this book contains an extensive set of problems. Some of these problems fill in technical details required in various proofs; indeed, the reader who reads Tate's thesis alongside Chapter 7 will find the answers to several of these problems. Other problems develop various aspects of the theory more completely. For example, one set of problems describes Pontryagin duality for finite groups, while another guides the reader through a proof of the Tchebotarev density theorem.

One consequence of Tate's approach to L-functions as presented in this text is that it hides the role of theta functions that are so central in Hecke's approach. This is interesting, considering that the modern perspective on L-functions, arising out of the Langlands program, argues that functional equations for L-functions arise ultimately out of automorphic forms such as theta functions. Consequently, a reader of this text should not entirely forget about Hecke's proof.

Fourier Analysis on Number Fields is a textbook with fairly specific, but highly worthwhile goals. Its most distinctive feature is the thoroughness with which it treats the analytic background to Tate's thesis. Anyone interested in number theory, automorphic forms, or harmonic analysis will find this book stimulating and worthwhile reading.

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