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Rational curves on algebraic varieties, by János Kollár, Springer, Secaucus, NJ, 1996, viii+320 pp., $139.95, ISBN 3-540-60168-6

Two papers of Mori in 1979 and 1982 [M1], [M2] used deformations of curves to give magnificent new results in the classification of varieties. Kollár's book sets itself the task of working out these arguments and their subsequent developments at a reasonable level of detail and technicality. The book is an irreplaceable source of information for many recent topics in algebraic geometry. It contains in particular a reworking of Mori's treatment of the existence of rational curves on Fano varieties, of Mori's fundamental Theorem on the Cone, and an extended treatment of the notion of rational connectedness that has developed over the last 15 years into the modern replacement for the old question of rationality.

Some sections of this book certainly take perseverance on the part of the reader before yielding up their delights; the first two chapters in particular are certainly harder than anything else in the book, and the reader should skim through these briefly and lazily (thus emulating the experts), but work seriously at some of the exercises later in the book. After some philosophy and simple-minded introduction, this review works backwards through the book, starting with the new ideas and applications, and finishing with some remarks to encourage the reader through the more technical early chapters.

The second printing of the book (1999) contains corrections to Chapter I, Theorem 1.7.2; and Chapter IV, Lemma 4.15.2; and patches up a few spelling mistakes.

1. Two different ways of studying varieties

Algebraic geometry in higher dimensions is never really simple or intuitive. Philosophically, most work in the subject can be viewed as trying to by-pass the higher dimensions of a variety V in one way or another. Let me discuss two approaches, the old and the new:

(A) Cut V by hypersurfaces (that is, set function = constant), or choose a suitably large supply of rational functions to embed V into a bigger ambient space $\mathbb{P}^n$, thus providing many such sections.

(B) Study V by mapping curves to it, preferably embedding rational curves.

Since divisors and curves are subvarieties of codimension 1 and of dimension 1, you can think of these as "dual" approaches.

(A) is the theory of divisors and linear systems, traditional from the late 19th century. A *divisor* is a codimension 1 subvariety, viewed as a potential locus of zeros or poles of a rational function or as a hyperplane section. Most of algebraic geometry up to the time of Mori was dominated by this approach: the Riemann–Roch theorem, the Kodaira vanishing and embedding theorems, classification of varieties by the size of the pluricanonical linear systems $|nK|$ and so on. For example, an extreme case of the classification of varieties is when the canonical system K_V is ample, so that the variety has an embedding by canonical divisors. The opposite case, called "adjunction terminates", is when there is an ample divisor

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D on V such that $D+K_V$ is not even effective. In this case, in the classical theory, one expects to be able to cover V by rational curves; however, in dimension ≥ 3 the theory of linear systems seems to offer few clues as to how to prove this kind of thing.

(B) is the new theory initiated by Mori and the main topic of this book. It attacks the problem of curves on varieties in a number of novel ways. Since the methods are well adapted to dealing with varieties having many rational curves, it is mostly useful if X is covered by rational curves, that is, *uniruled* (see Section 4).

2. Introduction via baby cases

A simple-minded discussion of rational curves on varieties puts the general aspirations of the theory into context. Suppose we want to study a nonsingular projective variety V of dimension n; for definiteness, please bear in mind the case $n=3$. Suppose that $L \subset V$ is a nonsingular rational curve (that is, $L \cong \mathbb{P}^1$) and we want to see L move in a family of curves. This question is studied in many classical cases, often by completely elementary methods, and we can summarise the results by saying that

(1) in good cases, L moves in a nonsingular family of dimension
$$\deg T_V\big|_L + n - 3 = c_1(V)[L] + n - 3 = -K_V L + n - 3.$$

Here $T_V\big|_L$ is the restriction to L of the tangent bundle to V. Its degree is interpreted in topology as evaluating the first Chern class $c_1(T_V)$ of the tangent bundle T_V, or in algebraic geometry as intersection number with the canonical class, the divisor class of the highest exterior power of the cotangent bundle $\bigwedge^n(T_V^*)$. *Exercise 1:* Prove (1) under the reasonable assumption that the family of curves $\{L\}$ is nonsingular and has tangent space $H^0(N_{V|L})$. [Hint: $N_{V|L}$ is a rank $n-1$ vector bundle on $L \cong \mathbb{P}^1$, and its degree comes from the restriction exact sequence
$$0 \to T_L \to T_V\big|_L \to N_{V|L} \to 0,$$
so that this follows easily from Riemann–Roch.]

Moreover, if L is not actually given, but only its class in $H_2(V,\mathbb{Z})$, the same kind of studies lead (in good cases) to the conclusion that curves L exist. *Exercise 2:* Prove that a hypersurface $V_d \subset \mathbb{P}^{n+1}$ contains straight lines if $d \leq 2n-1$, and that if V_d is general these form a nonsingular family of dimension given by (1). Moreover, if $d \leq n$, there is a straight line L through every point of V_d. [Hint: See Shafarevich, *Basic algebraic geometry*, Chapter 1.]

Another useful case for forming intuition: a line pair (that is, a reducible curve C isomorphic to the singular conic $L_1 \cup L_2 \subset \mathbb{P}^2$) often moves in a family together with nonsingular conics. *Exercise 3:* A general quartic 3-fold $V_4 \subset \mathbb{P}^4$ contains intersecting lines. Use elementary coordinate geometry to prove that each line pair moves in a 2-dimensional family of conics (but only a one-dimensional family of line pairs). Compose and solve for yourself a more general exercise concerning a hypersurface $V_d \subset \mathbb{P}^{n+1}$ containing line pairs that move out to nonsingular conics.

The hard parts of this book provide precise techniques (deformation theory, Hilbert and Chow schemes, etc.) to manipulate curves on varieties; see §7 later in the review for a brief discussion. Many readers will get a lot out of the book by taking the precise results on trust *en première lecture*; the above remarks on dimension counting, while admittedly rushed and inadequate, provide some intuitive background.

3. Chapter V on Fanos

Chapter V on Fano varieties is probably the easiest of the book and has the most rewarding applications: a projective (nonsingular) variety X is *Fano* if its anticanonical divisor $-K_X$ is ample. This class contains many of the most common varieties in applications: hypersurfaces of low degree in $\mathbb{P}^{n+1}$, projective homogeneous spaces or hypersurfaces of low degree in them, simple cases of moduli spaces, etc. After discussing examples, V.1–4 runs through standard and more advanced topics concerned with rational curves on Fanos, mainly due to Mori, Miyaoka and Kollár: as you would expect from the naive dimension count of §2, a Fano X contains rational curves C with degree $-K_XC \leq n+1$ and is rationally connected; these results imply that all Fano n-folds form a bounded family. Characterisations of $\mathbb{P}^n$ in these terms are given, including Mori's famous proof of the Hartshorne conjecture: a nonsingular n-fold X with ample tangent bundle T_X is isomorphic to $\mathbb{P}^n$. The crux of the proof is to find a rational curve $C \subset X$ with $-K_XC = n+1$ to play the role of a line in $\mathbb{P}^n$.

4. Chapter IV on rational connectedness

Chapter IV introduces and discusses the condition that a variety be *rationally connected*. For applications of algebraic geometry (say, to Diophantine number theory), the preferred case is a rational variety: an n-fold X is *rational* if there is a isomorphism $\varphi\colon \mathbb{P}^n \dashrightarrow X$ from the complement of a hypersurface in $\mathbb{P}^n$ to a dense open set of X, with both φ and φ^{-1} defined by rational functions (in other words, a dense open set of X has a rational parametrisation, like the singular cubic curves in the baby textbooks). Unfortunately, rationality seems to behave pretty badly in dimension ≥ 3. Varieties that *ought* to be rational (for example, because they have no holomorphic tensor forms or are the surjective image of a rational variety) need not be. Despite immense and prolonged effort, no-one has been able to come up with a convincing criterion for rationality. There are strong reasons for suspecting that rationality and unirationality are not invariant under small deformation in dimension ≥ 4.

Two weaker conditions seem to be more amenable to a systematic treatment: we say that a variety X is *uniruled* (respectively *rationally connected*) if it contains enough rational curves C so that one passes through every point of X (respectively, one through every pair of points). The notion of rational connectedness (due to Miyaoka, Mori, and Kollár, and independently to Campana) is a reliable alternative to rationality and is the ultimate *raison d'être* of the whole book. Chapter IV is a systematic round-up of what is known about uniruled, rationally connected and related notions. For a variety X, we have the conjectural characterisation

$$\kappa(X) = -\infty \iff X \text{ uniruled} \iff \begin{array}{c}\exists \text{ a covering family of} \\ \text{curves } C \text{ with } K_XC < 0,\end{array}$$

which is proved in dimension ≤ 3 and char 0 (proved *a posteori*, that is to say, using the sum total of the results of Mori theory). Similar remarks apply to the characterisation of rationally connected in terms of holomorphic tensors. Under mild conditions (nonsingular and char 0), a uniruled variety is covered by rational curves of reasonably small degree, and a chain of rational curves smooths out to an irreducible rational curves so that rationally *chain* connected implies rationally connected.

The definition of maximal rationally connected (MRC) fibration is the higher dimensional analog of the distinction between rational and ruled surfaces. An MRC fibration is a fibre space $X \to Z$ with rationally connected fibres and dimension of Z as small as possible. The base space is conjectured not to be uniruled, so that the MRC fibration packs together all the freely moving rational curves of X into its fibres. Chapter IV concludes with a treatment of rational connectedness over an algebraically nonclosed field, aiming to discuss the generalisation of Tsen's theorem.

I confess that my enthusiasm for the interesting applications in Chapters IV and V has led me into various white lies concerning technical notions in the above discussion. The most serious of these is the pretence that the curves C under discussion are *subvarieties* $C \subset X$; in fact, as I discuss shortly, Mori's main technical invention is to study *morphisms* $\varphi\colon C \to X$ from a fixed curve (often $C = \mathbb{P}^1$, and we fix the image $\varphi(P_i) \in X$ of one or two points). A key point of Kollár's book, in use throughout Chapter IV, is the idea of a *free family* of rational curves, meaning a morphism $\varphi\colon \mathbb{P}^1 \to X$ whose deformation theory is unobstructed.

5. Bending-and-breaking and Mori's theorem on the cone

I turn back to Sections II.4–II.7 and Chapter III. For a projective variety X over $\mathbb{C}$, the homology and cohomology groups $H_2(X,\mathbb{R})$ and $H^2(X,\mathbb{R})$ are dual finite dimensional vector spaces. $H_2(X,\mathbb{R})$ contains the classes $[C]$ of algebraic curves, and H^2 the first Chern classes of line bundles (or the classes of Cartier divisors). It is traditional in algebraic geometry to introduce cycle class groups N_1X or N^1X as analogues of H_2 and H^2, thus achieving the same end by messing about with different equivalence relations on algebraic cycles: we write N_1X or N^1X for the vector space of algebraic 1-cycles or codimension 1 cycles (with coefficients in $\mathbb{R}$) up to numerical equivalence. The two advantages of this procedure are that it is purely algebraic and totally inscrutable. Section II.4 discusses the various algebraic cycles used in Mori theory and the equivalence relations and effectivity conditions on them. The key idea is the Kleiman–Mori cone $\overline{\mathrm{NE}}\,X \subset N_1X$, the closed convex hull of the classes of curves $[C]$. The behaviour of $\overline{\mathrm{NE}}\,X$ in the half-space $(K_X z < 0)$ is the subject of Mori's famous theorem on the cone, stated and proved in Chapter III along the lines of Mori's original proof, but with a number of refinements. This material is treated in several excellent colloquial surveys (see Kollár [Ko] and Corti and Reid [CR], Foreword), and I cut short the discussion.

Section II.5 studies bending-and-breaking: let X be a nonsingular projective variety, and $\Gamma \subset X$ a curve with $K_X\Gamma < 0$. We first normalise Γ to give a nonsingular curve C and a morphism $\varphi\colon C \to X$ and study the deformation theory of the morphism φ (from the *fixed* curve C). The normal bundle to φ is the cokernel of the differential

$$N_\varphi = \operatorname{coker}\Big\{ \mathrm{d}\varphi\colon T_C \to \varphi^*T_X \Big\}.$$

Its cohomology $H^i(N_\varphi)$ for $i = 0, 1$ controls the deformation theory of φ: for example, if $H^1(N_\varphi) = 0$, small deformations of φ are parametrised by a ball in

$H^0(N_\varphi)$. The condition $K_X\Gamma < 0$ is used to give the lower bound

$$\deg N_\varphi \geq -K_X\Gamma + 2g(C) - 2. \tag{2}$$

Thus if $K_X\Gamma$ is negative by some fixed amount, we conclude from Riemann–Roch that $\chi(C, N_\varphi) = h^0(N_\varphi) - h^1(N_\varphi)$ is positive, so that the morphism φ moves in a positive dimensional family. We can work in the same way with deformations of φ that fix one or two points of C, provided $K_X\Gamma$ is a bit more negative. For example, if $C \cong \mathbb{P}^1$ and $-K_X\Gamma > n+1$, the conclusion is that φ has a deformation family over a base curve B fixing two points of C (to help rigidify the automorphisms of $\mathbb{P}^1$) so that the image of C sweeps out a surface in X. If we try to extend the family $\varphi_B \colon B \times C \to X$ to the projective completion $\overline{B}$ of B, we find a rational map $\varphi_{\overline{B}} \colon \overline{B} \times C \dashrightarrow X$ that cannot possibly be a morphism near the sections $\overline{B} \times P_i$, because by construction these contract to points. It follows that a rational curve with $-K_X\Gamma > n+1$ moves in a family containing reducible curves.

Mori's treatment of the general case $K_X\Gamma < 0$ is much deeper: if we were in $\operatorname{char} p$, we could replace the morphism φ by its composite $\varphi \circ F_C^n$ with a power of the Frobenius endomorphism F_C of C to multiply $\deg N_\varphi$ by p^n, while fixing the genus of C; roughly, p^n times C then moves in a big enough family so that a rational curve of bounded degree breaks off. If we own rational curves of bounded degree in char p for infinitely many different p, the standard results that Hilbert schemes are bounded imply the existence of rational curves of bounded degree in char 0. This argument, although amazingly subtle, is really just a justification of the dimension count of §2.

The last two sections of Chapter III treat minimal models and the classification of surfaces, using Mori theory to provide the modern proof of Enriques and Castelnuovo's characterisation of ruled and rational surfaces. For a projective nonsingular surface, the direct conclusion from K_S not nef is a Mori extremal contraction $S \to B$. To go from this to the concrete conclusion that S either contains a -1-curve that can be contracted by Castelnuovo's criterion or is a $\mathbb{P}^1$ bundle over a curve or that $S \cong \mathbb{P}^2$ requires either topological input (we derive the class of a line from Poincaré duality) or a beautiful study of del Pezzo surfaces, which Kollár carries out in Section III.3, following Zariski.

6. Ingenious Frobenius

Sections II.6 and V.5 are clever digressions characteristic of the Kollár style, using char p arguments to get conclusions in char 0. Section II.6 on Kodaira vanishing is based on ideas of Ekedahl and Shepherd-Barron: starting from a counterexample to Kodaira vanishing, that is, an ample line bundle L on a variety X in char p with $H^1(X, L^{-1}) \neq 0$, a construction involving an inseparable cover gives a covering family of rational curves of small degree on X. This leads to a contradiction in many cases, and this idea can be used to prove Kodaira vanishing in char 0 or on a nonruled variety and in many other cases provides a good substitute for it. Section V.5 contains a new proof of the *irrationality* of many Fano hypersurfaces, an unexpected new twist due to Kollár: on a general Fano hypersurface in char p, the Frobenius endomorphism provides a destabilising subsheaf of the tangent bundle, and hence regular differential forms that would contradict separably uniruled.

7. Hilbert schemes, etc.

Constructions of algebraic geometry, such as subvarieties $Z \subset X$ of a fixed projective variety X, often depend on parameters. Grothendieck and Mumford's *Hilbert schemes* and Cayley's *Chow varieties* are two complementary approaches to the problem of making such parameter spaces into algebraic varieties. For many readers, the best prelude to the Hilbert scheme (and the functorial basis for moduli spaces) would possibly be a few weeks with Mumford's lectures [Mu] or Grothendieck's sketch [G]. The idea of the Hilbert scheme, as treated in Section I.1 of the book, is that Z is defined by its equations, forming a homogeneous ideal I_Z, and we can coordinatise Z by the ideal I_Z, viewed as a point in a Grassmann variety of vector subspaces of a polynomial ring; loosely speaking, a subvariety $Z \subset X$ is defined by the coefficients of finitely many polynomials. If we fix a Hilbert polynomial P describing the numerical data of the subvarieties Z, the corresponding Hilbert scheme Hilb_X^P is a projective scheme with a universal mapping property for parametrised families of subvarieties.

Once Hilb_X^P is defined and we know that it is bounded, it can be studied by infinitesimal methods as in Section I.2. The main technical result is Grothendieck's description of first order deformations of $Z \subset X$ and obstructions to extending them in terms of groups $\mathrm{Ext}_X^i(\mathcal{I}_{Z\subset X}, \mathcal{O}_Z)$, that in simple cases boil down to the cohomology H^0 and H^1 of the normal bundle to $Z \subset X$. In particular, applying this result successively gives lower bounds on the dimension of the components of Hilb_X^P, with the same flavour as the classical dimension counts. The first sections of Chapter II apply these ideas to the study of $\mathrm{Hom}(C, X)$, the scheme parametrising morphisms $\varphi\colon C \to X$ from a curve C. A Hom scheme is a particular case of a Hilbert scheme, since a morphism is defined by its graph viewed as a subvariety of the product $C \times X$. The aim is to establish the lower bounds mentioned in §5 on the dimension of $\mathrm{Hom}(C, X)$ at a morphism φ required to make the bending-and-breaking of Mori theory work. These sections are a detailed treatment of the ideas contained in two (rather difficult) pages of Mori [M1], pp. 595–597.

By contrast, Chow varieties study subvarieties $Z \subset X$ by a "jumping condition": we set up a big space consisting of incidence conditions that should happen to Z in codimension 1 and coordinatise Z by its *Cayley form*, the hypersurface where the accident happens. Sections I.3–I.6 set up Chow schemes precisely and study their relations with Hilbert schemes. It seems possible to me that the applications in Chapters III–V to higher dimensional varieties could be made independent of this, possibly the hardest part of the book. Be that as it may, Chow schemes have important advantages over Hilbert schemes in different contexts; the distinction is a bit like that between Cartier and Weil divisors. There is a comparison morphism from the Hilbert scheme to the Chow scheme (Theorem I.6.3). The idea that this correspondence is birational in important cases is exploited at several places in the literature on moduli spaces.

Chapter I and Section II.1–3 of Kollár's book confront head-on in heroic style many difficult technical issues for which no easy treatment should be expected. As Kollár says, it seems to be pretty well established that no treatment of deformation theory can be technically adequate while remaining comprehensible. Most algebraic geometers and singularity theorists know a little bit about deformation theory from experience of a number of examples (see, for example, [A], [R], [S]), but this knowledge is fundamentally inadequate for many purposes. Kollár makes a brilliant job of bringing the most important ones out into the open. Many of us will sleep or wave our arms more comfortably in our beds, respectively seminars, for knowing that Kollár has written up this technical material.

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