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4-manifolds and Kirby calculus, by Robert Gompf and András Stipsicz, Amer. Math. Soc., Providence, RI, 1999, xv + 558 pp., \$65.00, ISBN 0-8218-0994-6

1. Manifolds and handle decompositions

Kirby calculus has been passed down as an oral tradition for several years. Many topologists share a common mathematical ancestry, which includes legends of the handle calculus. These legends began when Smale introduced handle calculus in his proof of the high dimensional Poincaré conjecture [Sma]. Kirby calculus is a specialization of handle calculus to low dimensions. This specialization gives us a way to draw pictures of two, three and four-dimensional manifolds, together with a set of moves, which may be used to pass between any two descriptions of the same manifold.

Recall that a manifold is a space that is locally homeomorphic to $\mathbb{R}^n$. By Sard's theorem, the zero locus of any generic differentiable function will be a manifold. To demonstrate how Kirby calculus may be used to describe a 4-dimensional manifold, we will present some analysis of an elliptic surface via Kirby calculus. One model for the simplest elliptic surface is given below.

$$\begin{aligned} E(1) = \{([x:y:t],z) \in \mathbb{C}P^2 \times \mathbb{C} \mid x^2t + y^3 + (z^5-1)t^3 = 0\} \\ \cup\ \{([u:v:s],w) \in \mathbb{C}P^2 \times \mathbb{C} \mid u^2s + v^3 + (w-w^6)s^3 = 0\}, \end{aligned}$$

where the two factors of the union are identified on an overlap by $zw = 1$, $uz^3 = x$, $vz^2 = y$, $s = t$. A projection map is defined by $\pi : E(1) \to \mathbb{C}P^1$; $([x:y:t],z) \mapsto [z:1]$ and $([u:v:s],w) \mapsto [1:w]$. One may check that $E(1)$ is a 4 dimensional manifold using the implicit function theorem. The projection to $\mathbb{C}P^1$ gives $E(1)$ the structure of an elliptic surface. Every point in $\mathbb{C}P^1$ except the points $[1:w]$ with $w = w^6$ is a regular value of π. If $[z:1]$ is a regular value, there is a small disk, D_ε, about $[z:1]$, so that $\pi^{-1}(D_\varepsilon) = \pi^{-1}([z:1]) \times D_\varepsilon$.

To go further, we need the notion of a branched cover. A branched cover is a map $f : X \to Y$, that is a covering projection off a codimension 2 subset, called the branch locus. The branched cover is modeled on the map $g_n : \mathbb{R}^k \times \mathbb{C} \to: \mathbb{R}^k \times \mathbb{C}; (x,z) \mapsto (x, z^n)$. The map,

$$p : \pi^{-1}([z:1]) = \{[x:y:t] \mid x^2t + y^3 + (z^5-1)t^3 = 0\} \to \mathbb{C}P^1; [x:y:t] \mapsto [x:t]$$

is clearly a 3-fold branched covering with branch points $[1:0]$, $[\pm(1-z^5)^{\frac{1}{2}} : 1]$. Figure 1 is a picture of this cover. By inspection, the surface in this branched cover is orientable and has Euler characteristic zero. It is, therefore, a torus. Thus $\pi^{-1}(D_\varepsilon) \cong T^2 \times D^2$. We will now give a handle decomposition of $T^2 \times D^2$. When W_1 is an n-dimensional manifold with boundary, adding a k-handle to W_1 produces the new manifold, $W_2 = W_1 \cup_{S^{k-1}\times D^{n-k}} D^k \times D^{n-k}$. Here $S^{k-1} \times D^{n-k} \hookrightarrow (\partial D^k) \times D^{n-k} \hookrightarrow \partial(D^k \times D^{n-k})$, and we just pick any embedding of $S^{k-1} \times D^{n-k}$ into ∂W_1. The set $D^k \times \{0\} \subseteq D^k \times D^{n-k}$ is called the core of the handle. If we add handles to the empty set and end up with a manifold diffeomorphic to X^n, we will have a handle decomposition of X^n. Figure 2 is a handle decomposition of T^2.

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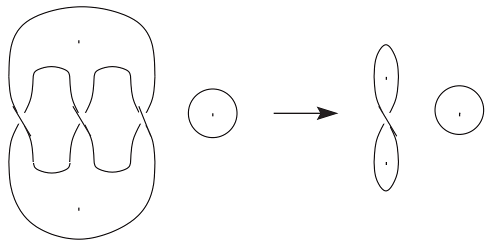

FIGURE 1. 3-fold branched cover

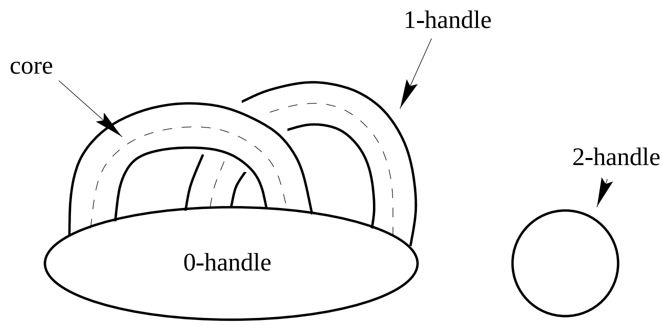

FIGURE 2. T^2

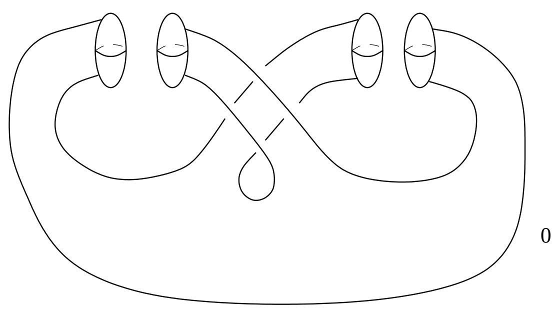

FIGURE 3. $T^2 \times D^2$

In order to get a handle decomposition of $T^2 \times D^2$, we just multiply the handle decomposition of T^2 by D^2. This is drawn in figure 3. The boundary of a 4-dimensional 0-handle is S^3 which we visualize as $\mathbb{R}^3 \cup \{\infty\}$. All of the other handles may be glued onto the $\mathbb{R}^3$. A 1-handle is a $D^1 \times D^3$ attached along an $S^0 \times D^3$ which is drawn as a set of balls in figure 3. A 2-handle is a $D^2 \times D^2$ attached along a $S^1 \times D^2$. To visualize the attaching map we draw $S^1 \times \{0\}$. The curve $S^1 \times \{1\}$ is parallel to $S^1 \times \{0\}$, but might link it some number of times. This linking number is called the framing and it is usually written next to the $S^1 \times \{0\}$.

A similar analysis will produce an excellent model of a neighborhood of one of the singular fibers. The singular fiber over $[1:0]$ is given by

$$\pi^{-1}([1:0]) = \{([u:v:s],0) \in \mathbb{C}P^2 \times \mathbb{C} \mid u^2 s + v^3 = 0\}.$$

The point $([0:1:0], 0)$ is a regular point of the singular fiber. The remainder of the singular fiber is an open cone on a trefoil knot. To see this, notice that the map, $h : [0,\infty) \times \{(u,v) \in \mathbb{C}^2 \mid u^2 + v^3 = 0, |u|^2 + |v|^2 = 1\}/\{(0,(0,0))\} \to \pi^{-1}([1:0]) - \{([0:1:0],0)\}$, given by $h(t,u,v) = ([t^3u : t^2v : 1], 0)$ is a homeomorphism. The domain is an open cone over $\{(u,v) \in \mathbb{C}^2 \mid u^2 + v^3 = 0, |u|^2 + |v|^2 = 1\}$. Let r be the positive root of $x^3 + x^2 - 1$. Simple algebra shows that the previous set may be described parametrically by $u = r^{\frac{3}{2}} e^{3i\theta + i\pi/2}$, $v = re^{2i\theta}$. Stereographic projection of this curve through the point, $u = 0, v = 1$ produces a trefoil knot. Alternatively, one may note that the set $\{(u,v) \mid |u| = r^{\frac{3}{2}}, |v| = r\}$ is a standardly embedded torus and that the base of the cone wraps twice around the v-direction as it wraps three times around the u-direction. Continuing in this manner, one can generate a handle decomposition of $E(1)$ and have a very clear understanding of the structure of the projection map.

Certainly drawing pictures of 4-dimensional manifolds is very cool, but is it useful? The answer is yes, it is actually useful. An example of a question that could not be answered without Kirby calculus is provided by the Scharlemann manifold. To construct the Scharlemann manifold, begin with the group of symmetries of an icosahedron. The inverse image of this group under the map $Sp_1 \to SO_3$ is called the binary icosahedral group. The quotient of Sp_1 by the binary icosahedral group is called the Poincaré homology sphere, Σ. The Scharlemann manifold is obtained by surgery on $\gamma \times pt$ in $\Sigma \times S^1$, where γ is any curve in Σ normally generating the fundamental group [Sch]. All known invariants of the Scharlemann manifold agreed with the invariants of $(S^3 \times S^1) \sharp (S^2 \times S^2)$. Recently, S. Akbulut proved that the Scharlemann manifold was diffeomorphic to $(S^3 \times S^1) \sharp (S^2 \times S^2)$ [A]. It took 18 years for him to find the right series of pictures to establish this difficult result. If there is some extra structure, proving that two manifolds are diffeomorphic via Kirby calculus could be fairly straightforward. However, in such situations, it will often be possible to establish the diffeomorphism without Kirby calculus. On the other hand, when there is no special structure, and Kirby calculus is the only tool available to establish a diffeomorphism, the result is likely to be very hard.

2. The book

A written record of the handle calculus legends is desirable for future generations. This book is a complete record of the folklore related to the handle calculus. Like any modern calculus text, this book is large (550 pages). It has many homework problems, and it begins with an algebra review.

Part 1 of the book consists of three chapters: "Introduction", "Surfaces in 4-manifolds", and "Complex surfaces". Part 1 may be summarized as a review of the algebraic topology of 4-manifolds and a review of 4-dimensional topology related to algebraic topology. While no one would consider gauge theory algebra, most would agree that the treatment of gauge theory given in part 1 of this book is just a review. For those already familiar with the theory of 4-manifolds, the first part of this book is a nice stroll down memory lane. Students and researchers who are not familiar with the theory of 4-manifolds might be scared away by the density of material contained in part 1. This would be a shame. This reviewer recommends that anyone wishing to learn Kirby calculus begin with part 2 of this book and refer back to the sections of part 1 as needed.

Some topics in part 1, such as the classification of indefinite unimodular forms, are covered in detail. Other topics in part 1, such as gauge theory, are only sketched with a list of the main results. D. Salamon has a large book in preparation on modern (Seiberg-Witten) gauge theory that will cover that topic in detail. Classical gauge theory (before basic classes) is covered in the book by Donaldson and Kronheimer [DK]. The book by Freedman and Quinn covers the theory of topological 4-manifolds [FQ]. The fact that whole books can and have been written on three disjoint topics in the study of 4-manifolds testifies to the rapid development of this field. It would have been difficult to fill one book about 4-manifolds twenty-five years ago.

Kirby calculus is the art of drawing pictures of low dimensional (especially 4-dimensional) manifolds and recognizing when two such manifolds are equivalent. The artwork developed in part 2 will help the Kirby calculus student master the algebra reviewed in part 1. For example, the end of section 4.4 together with section 4.5 will clarify the notion of the intersection pairing described at the beginning of the book.

The second part of this book begins with a description of handle decompositions and demonstrates how to represent some simple low-dimensional manifolds, including disk bundles over surfaces, and tori. The next topic covered is handle moves. Handle moves are used to pass between any two descriptions of the same manifold. Many manifolds may be expressed as the boundary of a higher dimensional manifold. The result of adding a handle to the boundary is called surgery when restricted to the boundary. Moves that can be used to pass between two surgery descriptions of a given manifold are described. In addition, many combinations of the basic moves (slam-dunks, Rolfsen twists, handle twists, etc.) are described.

This part of the book continues with the theory of handlebodies representing spin manifolds. The concrete constructions listed here nicely complement the homotopy theory of 4-manifolds from part 1. After a discussion of spin structures, the book describes more complicated 4-manifolds. This includes plumbings, Casson handles, complements of surfaces, and branched covers. By this point in the book, the student of Kirby calculus will be happy to see that answers to many of the exercises may be found at the end of the book.

The third and final part of the book is labeled as applications. It begins with a chapter on branched covers and resolutions. This chapter, in fact, provides many examples of things that can be done in the theory of 4-manifolds without Kirby calculus. Branched covers of algebraic surfaces are described from the point of view of algebraic geometry. Many different models of the elliptic surfaces are constructed without any handle diagrams. All of the various models of each surface are then shown to be diffeomorphic without reference to Kirby calculus. In addition, complex surfaces are constructed representing many of the possible values of the signature and Euler characteristic. This chapter is a nice follow-up to the previous chapter where branched covers are described via handle calculus. It will motivate a student to go back and understand the appendix on characteristic classes and the section on complex surfaces from the beginning of the book. The book by Harer, Kas and Kirby also covers resolutions and constructions of elliptic surfaces from a topological point of view; this book also provides handle decompositions of the elliptic surfaces [HKK]. The big difference between the book by Harer et al. and the book by Gompf and Stipsicz is size. The former might almost be considered as the *Cliffs Notes* of the latter. Only in this case, the short notes were written first. A revolution in the theory of 4-manifolds occurred in the interval between the appearance of [HKK] and this present book. The remaining chapters cover some of the latest topics in the theory of 4-manifolds.

The next chapter covers holomorphic and smooth Lefschetz pencils and Lefschetz fibrations. The best way to understand the definition of a Lefschetz pencil is to look at the proof that any projective surface $S \subset \mathbb{C}P^N$ may be realized as a Lefschetz pencil. The proof is to take a generic linear codimension 2 subspace A and consider the set of all hyperplanes containing A. The family of curves obtained by intersecting each hyperplane with S is the Lefschetz pencil. The subspace A will intersect S in a finite collection of points (called the base locus). Every curve in the pencil will pass through each of these points. A Lefschetz pencil with an empty base locus is called a Lefschetz fibration. After an abstract discussion of pencils and fibrations, the text returns to the cool artwork. The handle decompositions of Lefschetz fibrations are the largest covered in this book. One large handle decomposition was generated as the 2-fold branched cover of $S^2 \times S^2$ branched over the curve representing $(2n, 2m)$ in $H_2(S^2 \times S^2)$. See the work of Fuller for more in depth analysis of these monsters [F]. This chapter also contains the most complicated handle manipulations in the book. In the previous chapter a clean argument was given showing that two different descriptions of an elliptic surface were diffeomorphic. The complicated proof of the equivalence of two handle decompositions of an elliptic surface given in this chapter might leave one with the feeling that handle calculations should be avoided. The chapter ends with a discussion of the rational blow down operation defined by Fintushel and Stern [FS]. The discovery of the rational blow down is an excellent example of why someone would want to do handle calculations.

The coverage of Lefschetz pencils in a textbook is especially timely, given recent work in the theory of symplectic manifolds. Recently, Donaldson proved that a 4-manifold is symplectic only if it is a Lefschetz pencil [D]. The last two chapters of this book cover symplectic manifolds and the related concept of Stein domains. The symplectic fiber sum is discussed, and a proof that every Lefschetz pencil admits a symplectic structure is given. A summary of the relation between symplectic structures and Seiberg-Witten invariants concludes the second to last chapter.

A Stein manifold is a complex manifold that admits a biholomorphic embedding into some $\mathbb{C}^N$. A Stein domain is the corresponding object with boundary. Given the definition of a Stein manifold or a Stein domain it is perhaps a bit surprising that a Stein structure on a manifold may be completely described by a handle diagram, with only a small amount of additional structure. There can be no 3 or 4 handles, and all of the two handles must be attached along Legendrian links. This means that only one type of crossing is allowed and that no vertical tangents are allowed (cusps appear in place of vertical tangents). A Stein domain is the right generalization of a symplectic structure to manifolds with boundary. The boundary of a Stein domain is a contact 3-manifold. The very concrete description of a Stein domain provides one with a very practical method for computing the gauge-theoretic invariants of these manifolds. The theory is still developing, and the ultimate role of Stein manifolds in low-dimensional topology is still unclear.

There is one chapter between the chapter on Lefschetz fibrations and the chapter on symplectic manifolds that has not been discussed in this review. This chapter covers the cobordism classification of 4-manifolds, Akbulut's corks, and exotic differential structures on $\mathbb{R}^4$. If a pair of smooth 4-manifolds are homeomorphic, then there exists a contractible manifold with boundary in one member of the pair that may be removed and reglued with an involution to produce the other member of the pair. Such a contractible piece is called a cork. The coverage of exotic differential structures on $\mathbb{R}^4$ in this chapter is probably the most comprehensive coverage of exotic structures in print today.

The book *4-manifolds and Kirby calculus* is written very carefully. All of the mathematical statements are given in absolutely precise language, and the notation and terminology used are well chosen. The bibliography is very long and will guide the reader to the best sources for information on topics not covered completely in this book. Aside from the usual index and index of notation, the authors have also included an index of useful operations on handlebodies. Gompf and Stipsicz have produced a very comprehensive book on the topic of Kirby calculus. Most low-dimensional topologists will want to have access to this as a reference book, and any student who takes the time to carefully read this book will be rewarded with a thorough understanding of this fascinating field.

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