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Complex analysis, fundamentals of the classical theory of functions, by John Stalker, Birkhäuser, Boston, 1998, xi+228 pp., $92.00, ISBN 0-8176-4038-X

Complex analysis texts written in the early 1900's, mostly by British authors such as E. T. Copson [2] and E. T. Whittaker and G. N. Watson [4], devote considerable space in later chapters to what we call special functions. These include the gamma, the hypergeometric, the elliptic, the modular, and the zeta functions.

Paul Halmos [3] wrote an interesting discussion illustrating the neglect of these chapters on special functions when he was a graduate student at Illinois during the 1930's. Concerning Whittaker and Watson's book, Halmos writes,

> It has two parts: "The process of analysis" and "The transcendental functions". The second part (345 pages) is longer than the first, and I find it frightening. Its twelve chapters have titles such as "The zeta function of Riemann" (I am not yet trembling), "The confluent hypergeometric function" (now I am), and "Ellipsoidal harmonics and Lame's equation" (I am ready to flee to cohomology and ask for asylum). In the complex variable courses at Illinois in the 1930's Whittaker and Watson was frequently used, but, to the best of my knowledge, the second part was never entered.

The trend observed by Halmos continues today, with special functions receiving scant attention, and modern texts reflect this. In the past two decades, a large number of complex analysis texts have been written. These cover many interesting topics, but not special functions, with the exception of the gamma function or the zeta function.

Now, why should special functions be studied? The theory of special functions, developed by some of the greatest names in the history of mathematics and further studied by outstanding contemporary researchers, are essential to solving many important problems in mathematics and mathematical physics. Sometimes the applications of these functions can be surprising or unexpected; a dramatic recent example is de Branges's solution of the Bieberbach conjecture where the positivity of a specific hypergeometric function turned out to be important.

In fact, the properties of special functions can effectively motivate an account of complex analysis while at the same time providing tools which would be useful to a student of mathematics. It is good to see that Stalker's book reinstates special functions into the teaching of complex analysis by presenting such a well motivated account. His insights have also allowed him to put his presentation into a more logical order.

Stalker explains his reasoning in his preface:

> In "part I" [of Whittaker and Watson] we find the foundational material, the basic definitions and theorems. In "part II" we find examples and applications. Slowly we begin to understand why we read part I. Historically this is an anachronism. Pedagogically it is a disaster. Part II in fact predates part I, so clearly it can be taught first. Why should

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> a student have to wade through hundreds of pages before finding out what the subject is good for?

The first chapter of this book succinctly discusses special functions and their applications to the solutions of interesting and important problems. Among the special functions studied in this chapter are the gamma function, the logarithmic integral, the beta integral, the hypergeometric function and the zeta function. However, this is not a mere catalog of special functions; the author uses the functions to solve specific problems which then either motivate generalizations or introduce new problems. There is a high degree of logical coherence and organization in the topics chosen by the author. He starts with the gamma function and develops some of its properties which he then uses to derive an asymptotic formula for the probability of getting exactly n heads in 2n tosses. Immediately after this, the asymptotic formula is employed, a la Chebyshev, to derive inequalities for the number of primes less than a given number. The asymptotic formula in turn suggests the Stirling series for the gamma function which is then used to extend the domain of definition for the zeta function. This zeta function is used to obtain another theorem of Chebyshev on the distribution of primes. Similarly, the beta function is the basis for the development of the properties of the hypergeometric and Whittaker functions which are initially defined as integrals. The Jacobi and Laguerre polynomials arise in this context and their orthogonality is obtained.

Stalker's approach is informed by the history of mathematics, and in a number of places he points out how some concepts and results have been incorrectly attributed. Thus, Stalker calls the Euler-Maclaurin formula the Euler-Maclaurin-Jacobi formula. He points out that Jacobi found the error term as an integral involving the Bernoulli polynomial, whereas the two earlier authors found only the infinite series, which is often divergent or semi-convergent.

We may note here, as a supplement to Stalker's presentation, that Poisson had an expression for the error term about ten years before Jacobi. Poisson had used Fourier series – in fact, a particular case of the so-called Poisson summation formula – to find the remainder. Comparison of the results of Jacobi and Poisson yields not only the Fourier expansion of the Bernoulli polynomials restricted to $0 < x < 1$ but also the Euler-Maclaurin formula as a consequence of the Poisson summation. Poisson's work lacks rigor, however, since the proper foundation for the study of Fourier series was not given until a few years later by Dirichlet.

Though Stalker keeps history in mind he does not necessarily take the approach of the original researchers, making the text more efficient. Thus Stalker defines the gamma function by a definite integral and develops its properties, following Bohr and Mollerup, by means of logarithmic convexity – whereas Euler initially obtained an infinite product representation for this function and derived the definite integral only later. There is a small error on page 59 which states that Abel claimed without proof that $\sum_{n=2}^{\infty} 1/(n \log n)$ diverges. A complete proof of this result is contained in Abel [1].

One remarkable feature of Stalker's first chapter is that no complex variable methods are used. The author explains: "This may seem odd in a book on complex analysis, but it is not without reason. Real variable techniques, particularly those depending on convexity and Hölder's inequality, are often useful and worth knowing. They have their limits though, and we soon reach the point where further progress requires the techniques of complex analysis." Thus several topics and results introduced in the first chapter are discussed again after some complex variable theory is developed. As one example, consider this formula, mentioned in chapter 1:

$$F(a,b;c;x) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}F(a,1+a-c;1+a-b;(x)^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}F(b,1+b-c;1+b-a;(x)^{-1}),$$

where $F(a,b;c;x)$ is the hypergeometric function. This function is represented by a series which converges, in general, for $|x| < 1$. The author explains the difficulties involved in proving it and how the theory of complex variables would help. Stalker attributes this formula to Thomé, who presented the result in 1879.

(Actually, this formula appears in a 1835 paper of Kummer on hypergeometric functions. Kummer's comments show even more clearly the need for complex analysis. Kummer remarks that if $|x| < 1$, then the series on the left converges but the two series on the right diverge and the opposite happens when $|x| > 1$. So the formula appears to be useless. But, Kummer notes, the formula is meaningful since the hypergeometric function may be expanded by the series on the left for $|x| < 1$ and by the two series on the right for $|x| > 1$. To illustrate his point, Kummer specializes the values of a, b, c to arrive at

$$x\,F(1/2,1;3/2;-(x)^2) = \pi/2 - (x)^{-1}F(1/2,1;3/2;-(x)^{-2}).$$

The left side then gives the series expansion for the arctangent function when $|x| < 1$ and the right side gives the expansion for $|x| > 1$. It is clear that these ideas can be clarified by means of analytic continuation.)

The second chapter starts with contour integration and then moves to a discussion of analytic functions and Cauchy's integral formula. Since Stalker aims to interest his reader in complex analysis by illustrating its usefulness, he usually does not state the theorems in their strongest forms. Thus Cauchy's theorem is derived from Green's formula, which requires the assumption of the continuity of the derivative. Stalker introduces further complex variable methods for special topics such as the distribution of primes, Bernoulli polynomials and hypergeometric functions. Thus the Fourier expansion of a Bernoulli polynomial is obtained by writing it as a contour integral and then deforming the contour through a series of poles. Similarly, the theory of the hypergeometric function is extended beyond the point attained in chapter 1 by means of the Barnes integral, called by Stalker the Barnes-Mellin integral. (This is presumably because Barnes' work fits naturally into the framework of Mellin transform theory.) In this context, the author carefully explains how to apply the residue theorem and how to obtain bounds on integrals.

The third and final chapter deals with elliptic and modular functions, including a discussion of the theta functions, Jacobi elliptic functions and Eisenstein series. Stalker tells us that his exposition largely follows "Borchardt's notes from Jacobi's Berlin lectures, still in many respects the best introductory treatment available, but with much more extensive use of the calculus of residues." (C. Borchardt was a student of Jacobi and Dirichlet at Berlin in the 1840's. Later, he enjoyed the friendship of Weierstrass and was for many years (1855-1880) the editor of Crelle's journal.) Most introductory texts which include elliptic functions usually treat the Weierstrass functions, so it is nice to have an accessible treatment of the Jacobi functions since they are so useful in many problems. Moreover, a knowledge of Jacobi's functions comes in handy when reading Kronecker, who almost exclusively used these functions. The reader might find it useful to know that Copson [2] gives a fine treatment of both the elliptic functions.

The subject of complex multiplication of elliptic functions (or curves) has a very long history going back to Fagnano and Euler in the 18th century and it continues to be important today, a topic to which graduate students should surely be exposed. As an application of complex multiplication, the author gives Eisenstein's proof of the law of biquadratic reciprocity. This particular example is well chosen because it has historical as well as mathematical significance. Gauss discovered the phenomenon of complex multiplication as a youth. However, he published nothing on the subject and made only a passing mention of it in his *Disquisitiones.* Gauss also worked on the formulation and proof of the law of biquadratic reciprocity which led him to other discoveries, such as the theory of Gaussian integers. Elliptic functions, Gaussian integers and reciprocity all came together in Eisenstein's work which elicited the admiration of Gauss, not an easy man to impress.

Stalker's book covers a large amount of important and interesting mathematics, including several topics not available in introductory textbooks. The order of presentation and historical background make the material accessible and clear. The writing is succinct though well motivated and the author's familiarity with the nineteenth century mathematical literature enhances the presentation and provides valuable depth. Stalker's book should make an excellent text for self study or for a graduate or advanced undergraduate course in complex analysis. I recommend it highly.

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Ranjan Roy
Beloit College
E-mail address: royr@beloit.edu