BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 3, Pages 369–377
S 0273-0979(01)00906-5
Article electronically published on March 27, 2001

Quantum symmetries on operator algebras, by D. Evans and Y. Kawahigashi, Oxford Univ. Press, New York, 1998, xv + 829 pp., $200.00, ISBN 0-19-851175-2

Following a long and glorious tradition, all Hilbert spaces in this review will be supposed separable, and the text is peppered with little technical lies allowed for the sake of the story.

1. Mathematics

The term *operator algebra* will be used here to mean an algebra of bounded operators on a Hilbert space $\mathcal{H}$, which is closed under the adjoint operation $a \mapsto a^*$ defined by the formula $\langle a\psi, \eta\rangle = \langle \psi, a^*\eta\rangle$ for $\psi, \eta \in \mathcal{H}$. Two main classes of operator algebras are studied according to their completeness properties. C*-algebras are closed in the norm topology for operators on $\mathcal{H}$ and von Neumann algebras are closed in the strong operator topology (= topology of pointwise convergence on $\mathcal{H}$). One may harmlessly suppose that a von Neumann algebra contains the identity operator. Commutants of self-adjoint sets of operators are a rich source of von Neumann algebras. By the commutant of a set $\mathcal{S}$ of operators we mean the set of all operators on $\mathcal{H}$ which commute with every element of $\mathcal{S}$. Commutants are automatically strongly closed. A fundamental theorem of von Neumann in [25] asserts that the commutant of the commutant of a *-algebra is the same as its strong closure, so von Neumann algebras could alternatively be *defined* as commutants of self-adjoint sets.

Although strictly speaking a von Neumann algebra is a C*-algebra , in fact the two are as different as chalk and cheese, as can be seen by their abelian forms where two well known theorems give a complete classification:

The Gelfand-Naimark theorem states that any abelian C*-algebra is isomorphic to the *-algebra $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space X.

The spectral theorem shows that an abelian von Neumann algebra is isomorphic to $L^\infty(X, \mu)$ for some σ-finite measure space (X, μ).

Thus the collection of commutative C*-algebras is large and complicated whereas the list of commutative von Neumann algebras is quite short: $L^\infty([0, 1], dx)$, $l^\infty(\Sigma)$ for some countable set Σ, and direct sums of the two.

Having disposed of commutative operator algebras, let us consider those as non-commutative as possible. For C*-algebras these are the simple ones (i.e. no two-sided ideals), and the most non-commutative von Neumann algebras are the "factors" which are by definition von Neumann algebras whose centres contain only the scalar multiples of the identity. The most obvious simple C*-algebra is the algebra of all compact operators on $\mathcal{H}$ and the most obvious factor is the algebra of all bounded operators on $\mathcal{H}$. The first really interesting simple C*-algebra is found by taking two unitaries u and v on $\mathcal{H}$ such that $uv = e^{2\pi i\theta} vu$ for some irrational number θ. It is not hard to show that the isomorphism class of the C*-algebra generated by u and v depends only on θ and not on the particular choice of u and

2000 *Mathematics Subject Classification.* Primary 46Lxx, 81T75, 81T45, 81T05, 81T40, 81T08, 57R56.

v. This C*-algebra is thus simple and is known as the "irrational rotation algebra" A_θ. The first really interesting factor is obtained by taking a discrete group Γ and considering the strong closure of the algebra generated by the unitary group representation on $\ell^2(\Gamma)$ given by (left) multiplication on the group. This algebra, often written $U(\Gamma)$, should be called the von Neumann algebra of the group, for if Γ were finite, it is just the group algebra $\mathbb{C}\Gamma$. Then the centre of $U(\Gamma)$ consists of functions on Γ which are constant on conjugacy classes. Once Γ is infinite we have the possibility that its only finite conjugacy class is that of the identity. Thus since the operators must act on ℓ^2, the centre is reduced to multiples of the identity and we have a factor.

The contrast between von Neumann and C*- algebras could not be more glaring than what happens when one tries to decompose an algebra into its elementary constituents - factors on the one hand and simple C*-algebras on the other. In finite dimensions the Wedderburn theory applies and any operator algebra is the direct sum of matrix algebras, the sum being indexed by the minimal idempotents of the centre. Von Neumann generalised this theory beautifully to von Neumann algebras in [23]. One needs only to replace the direct sum by a "direct integral" over the measure space (X, μ) for which the centre of the algebra is $L^\infty(X, \mu)$. Algebra elements then become L^∞ sections of a measurable bundle over X. Thus the study of von Neumann algebras is reduced to factors. For C*-algebras the situation is vastly more complicated. The centre does not give precise information about the ideal structure; the ideals may sit in a very complicated fashion, and even if they don't, there may be topological obstructions to continuous sections. On top of this the space of simple C*-algebras is a big mess, even up to abstract isomorphism. (For instance the algebras A_θ are isomorphic exactly when two values of θ are in the same orbit under $SL(2, \mathbb{Z})$.) This is just what one would expect from the commutative situation - noncommutative general topology will not be any simpler than ordinary general topology! By the same reasoning one might ask if there is a straightforward classification of factors. The answer is yes and no as we shall now describe.

The first classification of factors is into types I, II and III. Let M be a factor. M is of type I if it is isomorphic to the algebra of all bounded operators on some $\mathcal{H}$. M is of type II_1 if it is infinite dimensional and admits a trace functional $tr : M \to \mathbb{C}$ such that $tr(ab) = tr(ba)$ for $a, b \in M$. M is of type II_∞ if it is not of type II_1 but contains an element p so that pMp is a type II_1 factor. M is of type III if it is neither of type I nor II.

Factors of all types were shown to exist in [24]. (The factors we have called $U(\Gamma)$ are of type II_1.) Just as remarkably, more than one type II_1 factor (up to isomorphism) was constructed in [17], and in [16] uncountably many were shown to exist and the classification of factors is not at all straightforward. That is the bad news.

Now the good news. A von Neumannn algebra is called *hyperfinite* if it contains an increasing dense sequence of finite dimensional *-subalgebras. In [17] it was shown that there is a *unique* hyperfinite II_1 factor. (It can be realised as $U(\Gamma)$ where Γ is the group of all finite permutations of $\mathbb{N}$.) Surprisingly, extending this uniqueness to type II_∞ was extremely difficult and achieved finally by Connes in [2]. In his thesis [3] Connes had already given a finer classification of type III factors into types III_λ with $0 \leq \lambda \leq 1$, and results of Krieger and Connes, using [2], showed uniqueness in the $0 < \lambda < 1$ case and a classification in the III_0 case.

The hyperfinite III_1 factor was shown to be unique by Haagerup in [13]. Thus one can list all hyperfinite factors. This very positive result is *the* noncommutative extension of the classification of measure spaces. Connes has poetically said that it gives us a beautifully bound blank book into which we can now enter many pages of mathematics.

A C*-algebra satisfying the analogous condition to hyperfiniteness (one containing an increasing *norm-dense* family of finite dimensional *-subalgebras) is called an AF C*-algebra. Bratteli introduced a useful diagrammatic description of AF C*-algebras in [1] as well as a complete classification in principle. A satisfactory classification in terms of "dimension groups" was achieved by Elliott and Effros, Handelman and Shen - see [7], [6]. Though highly useful, this work is a lot easier than the corresponding von Neumann algebra results. A deeper aspect of C*-algebras that inhabits the same terrain is the work on amenable C*-algebras (ones whose commutant is hyperfinite in any representation). They are not classified, but K-theory has proven to be a powerful weapon here.

2. Physics

Why should anyone be interested in operator algebras?

It is a subject requiring a non-trivial technical background and which appears to be quite distant from what is often considered the central stuff of mathematics.

For the reviewer and a host of other workers in the field, the motivation comes from physics. Hilbert space is the first entry in the manual for the mathematical understanding of the universe on a small scale. Square integrability of the wave function, allowing the probabilistic interpretation of its absolute values, is mentioned in high school chemistry textbooks. The spectral theorem makes transparent the idea of realizing the underlying Hilbert space as wave functions in "momentum space" or any other space of observable parameters. The mathematical expression of the uncertainty principle - surely one of the most profound scientific discoveries of all time - is in terms of commutation relations among self-adjoint operators!

From this perspective the issue of what is central in mathematics appears quite different from the dominant mathematical paradigm. Operator algebras appear on the scene very early, especially von Neumann algebras, since the topology in which a von Neumann algebra is complete is exactly that of average values of observables, correlation coefficients and so on.

In ordinary quantum mechanics the von Neumann algebras one encounters are of type I. But as soon as one talks about quantum statistical mechanics, factors of types II and III are required. Putting together infinitely many particles requires taking limits first approached by von Neumann in [26] and now best understood in terms of various von Neumann algebra completions of C*-algebras. Quantum field theory, whatever the beast may be, certainly retains Hilbert space foundations, and we can expect to see factors in all their splendour as the mathematical receptacle for the fields. The "algebraic quantum field theory" of Haag, Kastler and others (see [11]) is an attempt to approach quantum field theory by seeing what constraints are imposed on the underlying operator algebras by general physical principles such as relativistic invariance and positivity of the energy. A von Neumann algebra of "localised observables" is postulated for each bounded region of space-time. Causality implies that these von Neumann algebras commute with each other if no physical signal can travel between the regions in which they are localised. The algebras act simultaneously on some Hilbert space which carries a unitary representation of the Poincaré (=Lorentz plus 4-d translations) group. The amount of structure that can be deduced from this data is quite remarkable.

3. The book's title

This, then, is the setting for the book under review. Operator algebras are the objects one is trying to understand and for good reasons. What about the term "quantum symmetry" in the title of the book? Symmetry is a fundamental notion whose mathematical expression has become the presence of a group of transformations preserving some structure. In the operator algebra world this group becomes a group of automorphisms of the relevant operator algebra. This is the meaning of "symmetry" in our context, but the word "quantum" is new. Other lines of development, such as the (perturbative) theory of quantum groups, have also led to a concept of symmetry more general than that afforded by groups.

To see one way in which quantum symmetry arose entirely within the theory of factors, we turn to Galois theory. Groups themselves came from permutations of the roots of polynomial equations which became the Galois group of a field extension. The degree of a field extension is the dimension of the big field as a vector space over the small one. But in von Neumann algebras there is also a notion of dimension - of a module over a II_1 factor (the "coupling constant" of Murray and von Neumann). Indeed this is the first seductive aspect of operator algebras, as this dimension is a continuously varying real number. One can now simply copy Galois theory: given a subfactor $N \subseteq M$, define the degree of the extension to be the dimension of M as a left N-module. For historical reasons it is called the *index* of N in M, written $[M : N]$. The first surprise is the answer to the question: "What are the possible values of $[M : N]$?" To answer this we need to find lots of subfactors. Galois theory suggests looking at fixed points for group actions on M. A Galois-like theory for such subfactors was worked out in the 1950's [18], but that theory, no matter how one tries to fiddle with it, supplies only integer values for the index $[M : N]$. On the other hand the index is a von Neumann dimension, so we expect it to vary continuously. The intriguing answer to the question is that there is a discrete and a continuous part to the set of possible values. If $[M : N] < 4$, then the index is necessarily one of the numbers $4\cos^2 \pi/n$ for some integer $n \geq 3$, whereas all numbers ≥ 4 can occur. If we were to think of these subfactors in Galois theory terms, they correspond to finite "groups" of real order!

4. What's in the book

The book is over 800 pages long. Roughly one third is devoted to an introduction to operator algebras, one third to matters motivated by physics and one third to subfactors. The introduction covers standard material mostly in the usual fashion with the exception of the Murray-von Neumann coupling constant, which is approached as the dimension of a Hilbert space as above. All properties of the coupling constant may be obtained in a way that is much easier than that of Murray and von Neumann. The physics part of the book is somewhat uneven. It consists of a highly detailed C*-algebraic look at the two dimensional classical Ising model of magnetism on the one hand, and a loosely constructed look at conformal field theory on the other - abandoning all attempts at mathematical rigour and systematic exposition. Even a superficial explanation of conformal field theory would be almost as long as the one given in the book itself, so we will not attempt it here.

Ocneanu has shown that subfactors (of finite index and depth) are equivalent to Topological Quantum Field Theories and so give a wealth of unitary representations of mapping class groups and braid groups. (Shown to contain faithful finite dimensional unitary representations of the braid groups in work of Krammer and Bigelow.) Unfortunately Ocneanu has been rather slow to publish his work. This book attempts to fill that gap in the literature in the section on subfactors. Unfortunately the going can be quite hard and the writing somewhat cryptic. But it has the great benefit of existence.

5. Subfactors

We have already explained how subfactors arose from the Galois theory of II_1 factors. The object which extends the notion of a finite group came from the proof of the "$4cos^2\pi/n$" theorem referred to above. This exceedingly rich structure has been studied from many points of view and came to be called by several names: "standard invariant", "tower of relative commutants", "paragroup", "λ lattice", "C*-tensor 2-category with conjugates" and "planar algebra", all of which are equivalent under suitable restrictions and lead to generalisations in different directions. Before going on to describe the animal in more detail let me point out one of the greatest surprises of all to come out of the theory. Finite groups are finite systems containing finitely many symmetries. But finite index subfactors may actually contain *infinitely* many symmetries. Indeed any compact Lie group, or any finitely generated discrete group, is completely captured by a finite index subfactor (or its standard invariant). To obtain the structure corresponding to *finitely* many quantum symmetries, one needs to impose the condition of "finite depth", which we will soon define.

The simplest access to the standard invariant is to consider a factor M as a bimodule over the subfactor N. It is then natural to try to decompose the kth tensor power $M_k = M \otimes_N M \otimes_N ... \otimes_N M$ (see [14]) into irreducible $N-N$, $N-M$, $M-N$ and $M-M$ bimodules. If $[M:N] < \infty$, this decomposition works purely algebraically and involves only finitely many irreducible bimodules for fixed k. Bimodule intertwiners between these bimodules form what Ocneanu dubbed the "paragroup" and can be made completely combinatorial by the choice of a basis. The "principal graph" is the graph whose vertices are the irreducible $N-N$ and $M-N$ bimodules and edges connect vertices according to the obvious prescription dictated by restricting the left action of M to N. N is said to be of "finite depth" in M if the principal graph is finite. The original approach was to define an algebra structure on M_k and to consider the finite dimensional algebras $V_k = Hom_{N-M_k}(M_k)$. This was the "tower of relative commutants". Finite depth is then the condition that the generating function $\sum_{k\geq 0} dim(V_k)z^k$ be rational.

Here are a few facts about standard invariants, beyond which little is known:

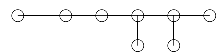
(i) Quantum groups $U_q(\mathcal{G})$ for $q \in \mathbb{R}^+$ or q a root of unity give subfactors for each representation of $\mathcal{G}$, the index being the square of the quantum dimension of the representation. The principal graph can be calculated [22],[28],[29].

(ii) The index of a finite depth subfactor is the square of the norm of the adjacency matrix of the principal graph, hence a totally real algebraic integer, largest among its conjugates [10].

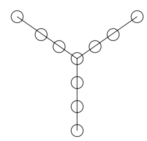
(iii) The algebras V_k defined above have a basis consisting of loops on the principal graph starting and ending at a privileged point.

(iv) If the index is less than 4, the principal graph is one of the Coxeter-Dynkin diagrams A_n, D_{2n}, E_6 or E_8. There is a unique paragroup for each diagram except for E_6 and E_8, which admit a pair of complex conjugate paragroups [20]. If the index is equal to 4, the principal graphs are the extended Coxeter-Dynkin diagrams, without restriction, and A_∞ and D_∞ [10].

(v) The smallest index of a finite depth subfactor greater than 4 is $\frac{5+\sqrt{13}}{2}$ (see [12]). There are two possible principal graphs:



and



6. Classification

There has been a distinct absence of analysis in our discussion of a subject based on Hilbert space! In fact analysis underlies the whole subject. In some sense it "makes the algebra work". For instance, the standard invariant itself has no apparent analytic content, but Popa's λ-lattice axiomatisation of the standard invariant arose as just what was needed to supply existence and uniqueness results for subfactors. The ideal result would be that to each standard invariant there is a unique subfactor of the hyperfinite II_1 factor. This is partly true. There is an amenability condition for a subfactor defined in terms of the random walk on the principal graph. For amenable subfactors (in particular finite depth ones) and standard invariants Popa has shown that the ideal result holds true. This is a deep theorem and implies among other things the Connes-Ocneanu classification of actions of discrete amenable groups on the hyperfinite II_1 factor (see [4], [19]). Outside the amenable world things go wrong in both directions. Using actions of free groups it is easy to construct families of subfactors with the same standard invariant, and an unpublished result of Popa implies that even the simplest case (the "Temperley-Lieb" algebra in planar algebra terminology) is not always obtainable from a hyperfinite subfactor. (But any λ-lattice does arise from a subfactor of a non-hyperfinite factor as shown by Popa in [21].)

Unfortunately the book under review does not supply any proofs or detailed discussion of Popa's classification results. This might be excused by the size of the book, but it certainly points out the need for a more comprehensive book on subfactors.

7. Physics again

You were supposed to be convinced by the argument that operator algebras are *the* mathematical framework for quantum theory. It would be nice to see some hard evidence for this!

Not long after the connection between subfactors and knot theory was found in 1984, papers began to appear exhibiting a connection with low dimensional quantum field theory - see [8]. In the 1970's Haag, Dopplicher and Roberts produced a theory of superselection sectors in [5] which came equipped with representations of the symmetric group for 4 space-time dimensional theories. These representations describe the "statistics" of the theory in the sense that bosons correspond to the trivial representation and fermions to the parity representation. In lower dimensions the disconnected nature of certain space-time regions forces the symmetric group to be replaced by the braid group. A broad spectrum of physicists were expecting exotic statistics in low dimensions, for instance in the fractional quantum Hall effect, and the term "anyon" was coined for such particles.

The mathematical object describing a superselection sector is an *endomorphism* of a factor with certain special properties - see [5],[15]. The image of the endomorphism is of course a subfactor whose index is known as the (square of the) "statistical dimension" of the sector. The special properties are enough to imply the existence of a unitary representation of the braid group inside the factor with the endomorphism acting as the shift on the usual braid generators.

From the subfactor point of view this picture was a revelation! All constructions of subfactors with irrational index values had previously been made, as presented in detail in the book under review, by combinatorial constructions whose point of departure was the proof of the original $4\cos^2 \pi/n$ theorem. Now at last there was a place to look for a more satisfying construction of subfactors - low dimensional quantum field theory. The one snag with the inspirational work on superselection sectors was that the actual mathematical objects were nowhere constructed. Their existence was assumed on the basis of overwhelming circumstantial evidence. The difficulties in constructing a mathematically sound quantum field theory are legend - see [9]. Thus it required quite some courage when Wassermann undertook the task of turning the physical ideas into mathematics - see [27]. He used Segal's framework of loop groups as the mathematical object supplying the quantum fields in one space-time dimension. He worked with the notion of bimodule or "correspondence" which Connes had shown to be equivalent to endomorphism. This was a crucial step as the inner product which Connes had used to define the tensor product becomes the 4-point function of the quantum field theory. Thus it was possible to use the Khniznik-Zamolodchikov equation of conformal field theory to calculate (rigorously!) the decomposition of the tensor product of bimodules. The structure of the corresponding subfactors, including the finite values of their indices and the braid group representations, comes out in the wash.

The book under review says very little about algebraic quantum field theory but does include a discussion of the sector/endomorphism picture.

8. Final comments

The field of operator algebras and their uses in physics is young and very active. Two of the main players in this game at the moment are subfactors and endomorphisms which supply a notion of "quantum symmetry". We can expect to see a lot more come out of this theory. Evans and Kawahigashi have written the first substantial book to treat this topic, and in spite of its drawbacks it should be useful to anyone wanting to learn the subject. At its best it provides clear explanations and good proofs. At its worst it should challenge the reader to get to the bottom of the topic under consideration.

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