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Quantum fields and strings: A course for mathematicians, Volumes 1 and 2, by Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten (eds.), American Mathematical Society, Providence, RI, 1999, (pt. 1) 723 pp., \$40.00, 0-8218-1987-9; (pt. 2) 778 pp., \$40.00, 0-8218-1988-7; (set) \$75.00, ISBN 0-8218-1198-3

Quantum theory began in an enigma about the structure of matter and radiation. This enigma pointed, in the hands of Planck and his successors, to a fundamental discreteness at the basis of physics. Such discrete behaviour would not be paradoxical in a fully discrete world, but it appeared at the time (and still does appear to many) that the world of our experience is well-approximated by a continuum. Certainly the classical phenomena of gravity, and electricity and magnetism appeared to be described by differential equations that modelled behaviour in a continuous world. Planck found that radiation needs to be quantized with a minimum allowed energy level, and the enigma was born. Einstein discovered that this same quantization hypothesis could explain the photoelectric effect, and atomic structure needed a new theory to allow the electrons to have stable orbits if they had orbits at all. One needed an explanation for the spectra of elements that would have the atoms emitting light at only certain characteristic frequencies. Bohr's theory of the atom gave the right numbers but was logically inconsistent. DeBroglie stepped into the picture and suggested that matter was accompanied by a wave and that this wave/particle duality of matter was the source of the elusive discrete. DeBroglie's suggestion explained the special orbits of electrons in atoms by a restriction due to the needed periodicity of his wave functions.

Werner Heisenberg discovered an algebraic approach to the atom where one no longer tried to visualize the atomic orbits. He took an approach that gave the coordinates for position and momentum the attributes of operators in a non-commutative calculus. Concurrently, Erwin Schrödinger found a wave equation to go along with DeBroglie's waves. That wave equation was in accord with the quantum behaviour of the hydrogen atom and with many other systems. In the context of the wave equation it was natural to replace physical observables by differential operators with special commutation relations. Soon it became clear that Heisenberg's matrix mechanics (of infinite matrices) and Schrödinger's wave mechanics were formally identical. The unification of these approaches led to a version of quantum mechanics where the states of the quantum system are vectors in a complex Hilbert space, and the observables are self-adjoint linear operators on that space. In this formulation all the physics rests in the structure of the observables.

The enigma remained, for it was still not clear how a particle could at the same time be a wave. One can see the dilemma clearly in the Schrödinger context, for there, without observation, the wave function evolves deterministically, but as soon as an observation is made the wave function is projected into an eigenstate. Observations are discontinuous interruptions of the smooth evolution of the wave

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function. And it is at the point of observation that the particulate appearance of matter and light emerges.

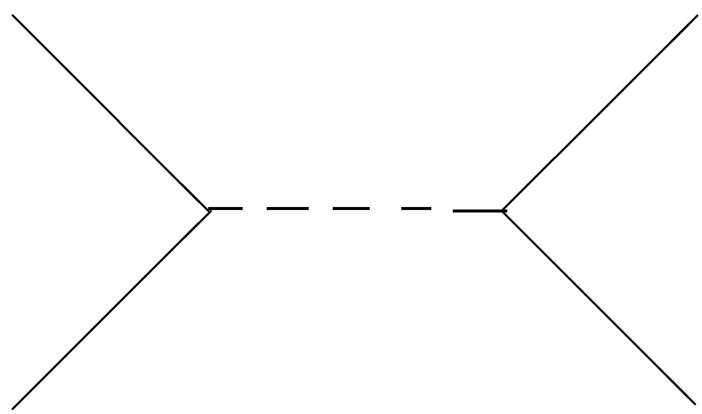
Electrons boil out of a hot filament and make their way toward a phosphorescent screen. Their journey is interrupted by a wall with two small separated slits. After many hits on the screen an interference pattern, ostensibly due to the two slits, is observed. This interference pattern is made up of a myriad of tiny dots, each a record of the collision of a single electron with the screen. How do the electrons "know" how to form the interference pattern? DeBroglie's answer was that the electrons really did fly in complex trajectories from the filament to the screen, but that these trajectories were governed by a "pilot wave", essentially the wave function of Schrödinger's equation. Later, David Bohm would take the DeBroglie theme and show exactly how it could be realized in the context of that equation of Schrödinger. But Heisenberg had made his theory in the context of a philosophy that would deliberately ignore what could not be observed. By a modern version of Occam's razor Bohr and Heisenberg suggested that it was not necessary to think of complex trajectories for the electrons, not even necessary to think of trajectories at all. Max Born put the cap on this model of logical positivism by interpreting the absolute square of the Schrödinger wave function as the probability of finding an electron (or other particle) if an observation is made. The wave function in the hands of these physicists became a wave of information, not a physical vibration, and the character of quantum mechanics became essentially statistical in a statistics where one no longer could point to any individuals that were tallied and counted.

Sometime later Dirac found a way to make a new version of the Schrödinger equation that was relativistically invariant, and this led deeper into the quantum mechanics of the electron and to the concept and experimental reality of anti-matter. With this deepening of a quantum mechanical approach to electrons came the need to quantize the electromagnetic field as well, to model the dual nature of light itself as particles and waves (the particulate nature of light as photons having been already verified by Einstein through the photoelectric effect). Field quantization meant the replacement of the smoothly varying electromagnetic field by a smoothly varying set of operators and commutation relations, in analogy to the more elementary commutators of the standard quantum theory. Thus quantum field theory was born and with it a host of physical and mathematical problems.

Almost as a concomitant to quantum field theory was Feynman's discovery that virtually all quantum problems could be formulated in terms of a summation over configurations of classical states (Feynman integrals and path integrals). This approach clarified the nature of the relationship of the classical and the quantum, but deepened the enigma. Now trajectories and classical fields were not banished, but Nature in the guise of the models promoted by the Feynman integrals used all possible trajectories and all possible fields. When all possibilities are integrated with appropriate complex number valued weightings, then most cancel each other out and the heart of what is left (the stationary phase approximation) is the familiar classical physics following the familiar classical laws. Surrounding that classical core is the halo of the quantum effects in exact analogy to the way ray optics and diffraction effects are related in the classical phenomena of light scattering. Applied to summations over fields, the Feynman integrals led to the famous Feynman diagrams, summaries of key contributions to the interactions that must be seen as sums over all possible events.

The sums themselves led to mathematical problems and physical problems. On the mathematical side the integrals do not always have an associated measure theory. A mathematician would tend to say that the integrals do not exist. A physicist would say that the integrals are hypotheses about the possibility of a calculation whose reality would be tested by its results. This is a pragmatic approach to mathematics: Assume the answer has a certain form and determine what sorts of answers can have that form. On the physical side, once it was a given that one had to sum over all possible interactions to have the state of (say) an electron, then that electron became surrounded by a cloud of (virtual) photons, indicators of the way the electron interacted with its own electromagnetic field. The summations diverged to infinity. Various "renormalization schemes" came into being to tame these self-generated infinities of quantum electrodynamics. All quantum field theories beyond a certain complexity require such renormalization, and the problem of making mathematical sense of these matters of integration and renormalization continues to the present day.

String theory was originally a corner of quantum field theory, born of observations related to the so-called "dual models" of Veneziano. In these models two classes of Feynman diagrams were intimately related.



View the diagram above in two ways: with time running vertically and space running horizontally, or with time running horizontally and space running vertically. In the first case you see two particles exchange a photon (the dotted line). In the second case you see two particles annihilate one another, producing a photon, and then the photon creates a pair of particles again. These quite different scenarios have related quantum amplitudes, and it was suggested that there should be a single mathematical scheme that would give rise to both (types of) calculations. Extended quantum particles were imaginable in the (more literal) strings of quarks that should arise to protect the separation of quarks of fractional charge. Nambu, Scherk and others in the 1970's suggested that the underlying mathematical structure could be seen through replacing the point (diagrammatic) models for the interacting particles by *closed circles* (the string!). The Feynman diagram becomes a surface topologically equivalent to a sphere with four holes punctured. The two forms of Feynman diagram become two of the topological forms of the single surface.

Along with replacing the particles by one-dimensional objects, the nascent string theory used analysis (of complex variables) related to the string surface (the world line of the string interaction) to relate the mathematics of the two different interaction pictures. The mathematics of this beginning conformal field theory was sufficiently compelling to keep people working at this attempt at unification. It must be understood that the string theory did not mean a return to a naive picture of an elementary particle as something like a classical one-dimensional loop, like a loop of macroscopic rope. It is a quantum string. But what would a quantum string be like? What would it be like to have a little vibrating string in the quantum domain? The answer comes from quantum field theory. A string is an example of a field with one dimension of space and values (the position of the string) in some other spacetime where the spatial dimension can be regarded at first as a matter of discussion. The motion of the string is a dynamical behaviour of this field. To quantize the string, one looks for a way to parametrize the field with operators and to identify the appropriate commutation relations in direct analogy with other quantum field theories. This gives rise to the Virasoro algebra and the conformal field theories associated with the string surface. From the mathematical point of view conformal field theory may be defined as a study of the Virasoro algebra and its representations and of their intertwiners. String theory is an outgrowth of quantum field theory. It does not start from nothing and it does not start from simple geometry, even though the string surface is an attractive image and a carrier of simplification for the physics.

The need for cancellation of anomalies in the string quantization shifts the embedding space to high dimensions. These extra dimensions may be real at the highly microscopic level of space. As the popular accounts tell us, the high dimensions of the superstring are rolled up in a microworld where vibrating strings lie in the heart of matter and radiation. These days, strings are generalized to membranes, and there is hope that the multiplicity of string theories will be unified in one magic matrix membrane theory emerging from the shadows of mathematical and physical analysis and including in its structure a full unification of quantum theory, high energy physics and general relativity.

They say that you should not judge a book by its cover, but that is not true for the two-volume set *Quantum fields and strings: A course for mathematicians*, edited by Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison and Edward Witten.

The wonderful cartoon on the cover of these volumes tells all. It consists of four quadrants, two labelled '68 and two labelled '98. In '68 there are two plates labelled "Physics" and "Mathematics" respectively. In '68 the physicists are busily contemplating a Feynman diagram and some highly indexed tensors on the blackboard. In '68 the mathematicians are conceptualizing a formula for the index of an operator in terms of the Chern character, some surfaces, some cohomology, some fundamental groups, an aura of differential geometry and differential topology. In '98 the cartoons are exactly reversed. The mathematicians puzzle over a Feynman diagram, and the physicists worry about topology, homology and differential geometry. What has happened that the tables have so turned? That is the story told in these volumes.

Here the reader will find accounts of physical structures by mathematicians, accounts of mathematics by physicists, and all manner of things in between. String theory and quantum field theory have such a rich mathematical structure that it has attracted the talents of both mathematicians and physicists, with many contributions to both mathematics and physics coming out of this mix. The books should certainly be on the shelf of every mathematician with an interest in connections with physics. By now Feynman integrals and Feynman diagrams have indeed infiltrated the mathematical camps. The integrals are still not necessarily defined, but the connections with rigorous mathematics are so strong in nearly all these cases that the testing ground for functional integrals has now shifted from the results of physical experiment to the results of proved conjectures emanating from the integrals. These books give a sweep of clear mathematical exposition for many of these issues. Here follows a quick description of the contents.

Volume 1 begins with a glossary of terms. You can look up "anomaly", "BRST", "supergravity" or "Feynman diagram". Curiously, "string" is not in the glossary. It is a good place for a glossary, right up front ready to excite and satisfy the reader's curiosity.

Part 1 of the book is entitled "Classical Fields and Supersymmetry". The first article is "Notes on Supersymmetry" – following lectures of Joseph Bernstein with notes by Pierre Deligne and John Morgan. This covers multilinear algebra, graded objects, supertrace, the Berezin integral, super manifolds and super Lie algebras. This article is followed by "Notes on Spinors" by Pierre Deligne, treating spinors and Clifford algebras. After this an article on "Classical Field Theory" by Pierre Deligne and Daniel S. Freed is a very clear mathematical treatment of classical fields and Lagrangians including electromagnetism, gauge theory and Yang-Mills theory, discussion of topological terms, Chern-Simons forms and the Wess-Zumino-Witten model. Then comes "Supersolutions" by Pierre Deligne and Daniel S. Freed discussing superspace, supersymmetry, sigma models and Yang-Mills theory. A summary of sign conventions occurs in the article "Sign Manifesto" by Pierre Deligne and Daniel S. Freed.

Part 2 of Volume 1 is entitled "Formal Aspects of QFT" and begins with "Note on Quantization" by Pierre Deligne. This essay discusses quantization in the context of the replacement of Poisson brackets by commutators and the mathematical general treatment that emanates from this key idea of Dirac. Then comes an "Introduction to QFT" by David Kazhdan giving the Wightman axioms and their consequences for free field theory, scattering theory and Feynman graphs. The next lesson is "Perturbative Quantum Field Theory" by Edward Witten, a passage through correlation functions, specific Lagrangians, perturbative expansion and Feynman diagrams, Feynman integrals, renormalization, critical dimensions (for renormalization) for sigma models, gravity, gauge theory, operator product expansions, scattering theory and asymptotic freedom. Then comes "Index of Dirac Operators" by Edward Witten with the heat equation and a path integral proof of the Atiyah-Singer index theorem for the Dirac operator, Dirac operators on the loop space LM of a manifold M, modular properties and circle actions. Then the book returns to basics with an "Elementary Introduction to QFT" by Ludwig Faddeev. This article starts with a clear description of observables and states, does path integral, quantization, harmonic oscillator, and begins quantum field theory, S-matrix, Lagrangians, with applications to electromagnetism, gravity and Yang-Mills theory. Part 2 ends with "Renormalization Groups" by David Gross and a "Note on Dimensional Regularization" by Pavel Ettingof. The article by David Gross discusses renormalization group, applications to phase transition, dimensional regularization, dynamical mass generation and symmetry breaking in the Gross-Neveu model and the Wilsonian renormalization group equation.

But we should not forget the homework problems. The first volume ends with the statements and solutions of the many problems that were distributed to the group during the special year (1996-1997) at the Institute for Advanced Study at Princeton. The problems are edited by Edward Witten with solutions by different members of the group.

Volume 2 begins with Part 3: "Conformal Field Theory and Strings". The first set of lectures is by Krzysztof Gawędzki on Conformal Field Theory. These cover conformal free fields, axiomatic approach to conformal field theory, perturbative analysis of two-dimensional sigma models, exact solutions of Wess-Zumino-Witten and coset theories, functional integrals, axiomatic approaches, Segal's topological geometric approach, sigma models, renormalization group and the Wess-Zumino-Witten model. Then come the lectures on "Perturbative String Theory" by Eric D'Hoker. These consist of an intuitive introduction to strings, the loop expansion, worldsheet action, free bosonic strings and the transition to conformal field theory and Virasoro algebra, spectrum, states, vertex operators, identification of the graviton in relation to the string, Chan-Paton rules, string amplitudes and moduli space of curves, Faddeev-Popov Ghosts and BRST quantization, strings on general manifolds, free superstrings, heterotic strings, supersymmetry and supergravity. This article by D'Hoker in the second volume should certainly be scanned first by any reader interested in the physical background and motivations for the mathematics that is discussed throughout this collection. The lectures on strings are followed by "Super Space Descriptions of Super Gravity" by Pierre Deligne and "Notes on 2d Conformal Field Theory and String Theory by Dennis Gaitsgory (explicating D'Hoker and making it much more abstract). And finally, "Kaluza-Klein Compactification, Supersymmetry and Calabi-Yau Spaces" by Andrew Strominger, a study of the Lagrangian arising from the heterotic string in 10 dimensions, the Kaluza-Klein theory, Einstein's equations, supersymmetry and Calabi-Yau manifolds.

Part 4 is entitled "Dynamical Aspects of QFT" and begins with "Dynamics of Quantum Field Theory" by Edward Witten. This covers dynamics of gauge theories, symmetry breaking, Goldstone's theorem, infrared behaviour, BRST quantization, sigma models, large N limit of sigma model to Grassmannians, Bose-Fermi correspondence and applications, two-dimensional gauge theory of bosons, Wilson line operator and confinement, abelian duality, solitons, 't Hooft loops and confinement, quantum gauge theories in 2 dimensions and intersection theory on moduli spaces, supersymmetric field theories, Landau-Ginzburg description of N=2 minimal models and quantum cohomology of Kähler manifolds, four-dimensional gauge theories, Yang-Mills and a description of the physical background to the Seiberg-Witten equations. Witten's article is followed by "Dynamics of N=1 Supersymmetric Field Theories in Four Dimensions" by Nathan Seiberg, covering the role of electric-magnetic duality in dynamics, relations with string theory, the possibility of experimental verification of supersymmetry, and topological quantum field theory based on supersymmetry.

This completes a survey of the contents. These two volumes constitute an extraordinary resource for anyone interested in theoretical physics and its relations with mathematics.

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