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Dynamics in one complex variable, Introductory lectures, by John Milnor, 2nd edition, Vieweg Verlag, Weisbaden, 2000, vii+257 pp., ISBN 3-528-13130-6

I am a Milnor fan. Although I first met Milnor in 1982, as a sophomore (now 37 years ago) I read *Topology from a differentiable viewpoint*, still in my view the best mathematics book ever written. It had a lot to do with why I became a mathematician, and what sort of mathematician. Other Milnor books followed: *Morse theory*, *Lectures on the H-cobordism theorem*, *Singular points of complex hypersurfaces*, each a turning point in my view of mathematics.

My undergraduate advisor, Raoul Bott, himself a great expositor, always said: to learn to write mathematics well, read Milnor and try to emulate his style. There aren't many who have Milnor's extraordinary ability to take a complicated subject and present it in such a luminous way that everything seems natural, easy and perfectly clear.

All this to say that I am by no means a detached and unprejudiced observer: I came to the book under review expecting to love it, and this expectation was fulfilled.

The field of complex dynamics: A brief history. Complex analytic dynamics has quite a long history, starting with the work of Schröder, Koenigs, Schwarz in the 19th century, and brought to a first flowering by Fatou and Julia in several long memoirs around 1920.

The subject then became dormant, with occasional but very significant publications by Cremer (1927, 1938), Siegel (1942), and Brolin (1965). During the 1970's, Guckenheimer and Jakobson published papers containing important new insights and techniques. Unfortunately, both papers contained serious errors; as a result they had much less influence than they otherwise would have had.

One anecdote to describe the dormant state of the field. Heinz-Otto Peitgen tells me that in the mid 1980's, he went to Sweden to interview Brolin, who had become the director of a technical high school. As far as Brolin knew, his thesis had been forgotten, and he was amazed to hear that on the contrary it was then (i.e., in 1985) viewed as a foundational work of an active new field and that a whole generation of mathematicians was learning the topic by reading his paper, in which he had put the works of Fatou and Julia into modern language and introduced potential theory into the field.

Since about 1980, the subject has undergone explosive growth. Why did the subject go to sleep? One reason is surely that during the period 1920-1975 differential equations and all associated dynamical questions lost the central position they had formerly occupied in mathematics (except in the Soviet Union). But there is another reason, which explains much of why the subject came back to life: computer graphics.

The field of complex dynamics: The modern rebirth. I have one telling anecdote to relate. In 1982, I was visiting Harvard, so was Mandelbrot, and we were both showing pictures of Julia sets and the Mandelbrot set. One day Ahlfors

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told me that in his youth, his thesis director, Lindelöf, had made him read the memoirs of Fatou and Julia, the recent prize memoirs of the French Académie des Sciences. Ahlfors told me that at the time, they struck him as "the pits of complex analysis." Further, he said that he only understood what Fatou and Julia had been getting at when he saw the pictures Mandelbrot and I were showing.

Today, it is hard to imagine how incomprehensible the subject must have been before computer pictures (and even harder to imagine how Fatou and Julia managed to write their papers). In any case, the arrival of computers and computer graphics transformed the field. The books by Mandelbrot and H. O. Peitgen containing pictures of Julia sets and the Mandelbrot set had a lot to do with the new popularity of the subject.

From a theoretical point of view, the main change in the field was the introduction of quasi-conformal mappings into the field by Sullivan, specifically in his proof of the *No Wandering Domains Theorem*. Further developments of this technique became the new field of *quasi-conformal surgery*.

Other important new tools were introduced, notably by Douady, Thurston, Shishikura, Yoccoz, Lyubich and McMullen.

Expository texts. It doesn't seem reasonable to review one book without speaking of the competition, especially when, as in this case, the contents of the books are so similar. In the early 1990's, four books on dynamics in one complex variable appeared:

A. Beardon, *Iteration of rational functions*, Grad. Texts Math. 132, Springer (1991);

L. Carleson and T. Gamelin, *Complex dynamics*, Springer (1993);

N. Steinmetz, *Rational iteration, complex analytic dynamical systems*, de Gruyter (1993);

and the volume by Milnor under review, which was published later but was available as a Stony Brook preprint since 1990 and evidently influenced all the others.

These books all cover roughly the same material: the decomposition of the Riemann sphere into the Julia set and the Fatou set, the local structure of periodic points, and Sullivan's classification of components of the Fatou set. These are all good books, which differ in emphasis and style, and each fills a different niche.

Beardon's book is more elementary than the others and makes a good text for an introductory graduate course. I know from experience that graduate students often like it a lot and find it considerably easier than any of the others. Neither is Beardon's book all that elementary. For instance, the presentation of Sullivan's theorem is very clear: that is where I learned Baker's remarkable simplification of Sullivan's original proof.

I would not consider using Carleson and Gamelin's book as a textbook in an introductory graduate class; it seems best adapted to professional analysts intent on learning complex dynamics from their own perspective. Certainly a graduate student might be put off by references to Stein's book [St] for the Calderon-Zygmund inequality, Gunning and Rossi [GR] for complex manifolds, or Hurewicz and Wallman [HW] for invariance of domain. But there is much here to admire that I have not seen anywhere else: the proof of Siegel's theorem adapted from KAM in Section II.6 and the non-triviality argument in the proof of Sullivan's theorem, in Section IV.1, for instance.

The book by Steinmetz is closest to Milnor's. In my experience students find it quite difficult to read, but there are many parts that I have read with great profit. His proof of Siegel's theorem, due to Rüssman, is very beautiful.

Parts of Milnor's book I especially like. Without explicitly quoting whole proofs, it is hard to convey exactly what makes Milnor's book so pleasant to read. I have used it in several graduate classes, and the students have consistently reacted favorably, although they find the book more difficult to read than Beardon's, and generally find the problems extremely challenging.

Below I will list some of the topics which strike me as especially well treated.

Section 2, on coverings and the Poincaré metric. This is a big topic, which is necessary because the contraction properties of the hyperbolic metric are the main tool of many proofs. But a complete treatment could sink the book, whereas too abbreviated a treatment would leave the reader in the dark. Milnor has found a selection of theorems, examples and comments which covers what is needed without getting bogged down.

Section 10, on parabolic fixed points. The treatment of the Fatou Flower in 10.5, and of the Ecalle cylinders in 10.6 and 10.7, is remarkably clear, and this is a subject which does not naturally lead to clarity.

Section 11, specifically the part on small cycles. I have not seen Theorem 11.13 elsewhere. Theorem 11.14 is also very nice; this extremely efficient proof of Siegel's theorem, due to Yoccoz, also appears in Carleson and Gamelin as Theorem V.1.4.

Section 12, the holomorphic fixed point formula. Milnor's text is the only one which treats this in detail, and he gets a lot of mileage from it in Sections 13 and 14.

The appendix on classical analysis is especially nice. This is the first place where I have seen a treatment of Jensen's inequality which really explains why it is true, and the same can be said of the Theorem of F. and M. Riesz.

Drawbacks. I don't see how the presentation of anything actually in Milnor's book can be improved. But his book, and all the others, omit topics which I feel are of central importance and which anyone who wishes to pursue complex dynamics will need to learn.

One is *Thurston's theorem on the topological characterization of rational functions* [DH3], [HS]. With this theorem, Thurston introduced Teichmüller theory into the subject. He also established a close connection between complex dynamics and symbolic dynamics, more particularly the theory of kneading sequences.

Another is *puzzles and tableaux* [BH], [HY]. This technique was introduced in [BH], and Yoccoz used it in his spectacular work on the local connectedness of Julia sets for quadratic polynomials which are not infinitely renormalizable and the local connectivity of the Mandelbrot set at the corresponding points.

Another is *renormalization* [McM]. None of the texts even mentions Feigenbaum or the influence of the many efforts to prove the Feigenbaum-Cvitanovic Conjecture. But these questions were behind much of the work of Sullivan, McMullen and others, and I think that at least the questions should be addressed, though in all fairness satisfactory answers were found only after the texts appeared.

Finally, though all the texts (except Steinmetz) make a stab at discussing parameter spaces, none puts nearly as much emphasis on this aspect of the theory as

I would put. The importance of exploring parameter spaces can hardly be exaggerated, and none of the texts reflects this. At the moment, the lecture notes [DH1] still seem to be the best source for this material.

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