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Alterations and resolution of singularities, H. Hauser, J. Lipman, F. Oort, and A. Quirós (Editors), Birkhäuser, Basel, 2000, xxi+598 pp., $108.00, ISBN 3-7643-6178-6

1. Introduction

The book contains a collection of articles by participants of the Working Week on Resolution of Singularities held at Obergurgl in Tirol, September 7-14, 1997. It is dedicated to Oscar Zariski, the founder of the school of algebraic geometry in the United States. During his long career as a mathematician he obtained ground-breaking results in algebra and algebraic geometry. Many years of his career were dedicated to the desingularization problem. His major achievements were the modernization of the classical theory of blow-ups and the proof of the existence of resolutions of singularities in dimension three.

The main focus of the book is on the substantial recent progress in the desingularization problem resulting from a rather powerful shift of the approach to the whole subject pioneered by J. de Jong.

In order to appreciate this change we need to consider the history of the subject. In geometry we are often dealing with objects which are locally similar at most points but exhibit exceptional behavior at a subset of points of smaller dimension. This subset is called the singular locus. More concretely, let us consider a simple topological version of the desingularization problem. Let X be a finite connected polyhedron of dimension d. Its smooth (or nonsingular) points are those which have a small neighborhood isomorphic to a ball. If we assume that X is a manifold without boundary and that every point of X lies on a simplex of dimension d, then a desingularization of X is a smooth manifold X' (without boundary) together with a surjective map $f : X' \to X$ such that f is an isomorphism on an open, everywhere dense subset $U \subset X'$. Using the desingularization (X', f) we can distribute the complexity concentrated at a singular point x in X over the subcomplex in X' containing the preimage $f^{-1}(x)$.

Unfortunately, in such a natural geometric setting a resolution does not exist in general! The simplest counterexample is given by a union of two copies of a cone over a real manifold which is not cobordant to zero. For example, an isolated unresolvable singularity would be a real cone over the real(!) fourfold $\mathbb{CP}^2$. Thus already in dimension 5 we can build a polyhedron which is a smooth manifold away from two singular points and which does not admit a resolution of singularities.

Amazingly enough, resolutions do exist for polyhedra associated with solutions of polynomial equations with coefficients in complex numbers. These very special subsets of complex affine (resp. projective) spaces are called algebraic varieties over the complex numbers. Easy examples of highly singular varieties are subvarieties of the affine space $\mathbb{A}^n$ given by generic homogeneous polynomials of degree ≥ 2 in n-variables. In spite of the complexity of the singularities which can appear on algebraic varieties, H. Hironaka, a student of Zariski, proved in the early 1960's the existence of a resolution in the strongest possible form. Precisely, for any

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projective variety X he constructed a smooth projective variety X' with a surjective (algebraic) map $f : X' \to X$ such that f is an isomorphism over an open dense subset of X. Here X' can be viewed, topologically, as a smooth even-dimensional manifold. Hironaka's construction satisfies several other properties:

- the map f is an isomorphism outside of the singular points of X;
- the preimage of the singular set is a union of smooth projective subvarieties of (complex) codimension one with transversal intersections (special smooth submanifolds of codimension two);
- X' and f are obtained through a sequence of standard operations (blow-ups of smooth subvarieties contained in the singular locus).

A blow-up, intuitively, replaces its *center* (an algebraic subvariety) by the set of its normal (complex) directions. In particular, one blow-up resolves the affine cone singularities described in the example above.

The initial proof of Hironaka was quite lengthy and complicated. Hironaka followed the classical path of blowing up the subvariety of most complex singular points. The problem which was encountered by Zariski and others was that up to dimension three the complexity of the singularity still has some geometric flavor, but in higher dimensions it lacks a geometrically intuitive characterization. Subsequently, Hironaka's proof and its logic have been substantially clarified (see, for example, the article of S. Encinas and O. Villamayor in the book under review, or the papers of E. Bierstone and P. Millman [2], and M. Spivakovski [5]). The general strategy is to introduce an appropriate function ϕ with values in a finite subset of a (lexicographically ordered) finitely generated semigroup (vectors with entries $\mathbf{N}$) and semicontinuous on X. The value of ϕ at each point reflects the complexity of the singularity. The maxima of ϕ are smooth inside the singular locus. Blowing up such points reduces the global maximum of ϕ on X.

This ultimate solution, which works for algebraic varieties over *any* field of characteristic zero, had a tremendous impact on the development of algebraic geometry. In practice, it substantially simplified computations of geometric invariants of algebraic varieties and provided a solid foundation for the subsequent advances in algebraic geometry. Bluntly put, it is one of the few universally useful mathematical results.

However, so far all attempts to extend the method to varieties defined over fields of finite characteristic or discrete valuation rings of mixed characteristic have failed. The original algorithm by Hironaka and its present modifications lead to loops: blow-ups don't diminish the maxima of ϕ.

For many years the spectacular success of the method of desingularizations by blow-ups developed by Segre, Zariski and Hironaka blinded, in a sense, other researchers in this field. It was so natural and powerful that other alternatives were abandoned or forgotten. This spell was broken by Johan de Jong in 1995.

Before explaining the approach of de Jong let us recall another important development in algebraic geometry. Independently of Hironaka's resolution of singularities, Deligne, Mumford, Knudsen and others were building the theory of stable degenerations of curves and compactifications of their moduli (natural parameter) spaces. Naturally, one wants to relate the invariants of the degeneration to invariants of the generic fiber. It is a standard trick in algebraic geometry to compute something on a maximally degenerated object and then extend the outcome (by continuity) to the smooth case. But there was no clear understanding of what justified such computations, and, not surprisingly, some of the computations were wrong. In the theory of moduli the main difficulty comes from the presence of automorphisms. Algebraic curves of a given genus degenerate into rather complicated one dimensional objects - combinations of singular curves with multiplicities. How to deal with this? The theory of stable curves and stable degenerations, in general, brought clarity to the subject. For every smooth family of curves $\mathcal{C}^* \to \Delta^*$ over a punctured disc there exists a unique relative compactification $\mathcal{C} \to \Delta$ (at least after changing the base by a finite cyclic covering of Δ^*). Moreover, the preimage of $0 = \Delta \setminus \Delta^*$ is a union of algebraic curves with normal crossing and very mild (nodal) singularities such that each smooth rational component of the preimage of 0 intersects the other components in at least two points. Such reducible curves (appearing as limits $\mathcal{C}_0$) are called semi-stable. Semi-stability is sufficient if one is interested in local properties of degenerations (or "coarse" moduli spaces). However, if one is interested in morphisms between degenerating families (or "fine" moduli spaces), one needs to introduce additional data to eliminate possible automorphisms of semi-stable curves. One approach is to fix a finite number of smooth points on curves and to keep track of these points. A second approach involves level structures. The corresponding moduli spaces are much more rigid. In particular, one has two crucial properties:

- for every family $\mathcal{C}^*$ of stable (punctured) curves over a (dense) Zariski open subset $\mathcal{B}^* \subset \mathcal{B}$ of a normal variety $\mathcal{B}$ there exists a generically finite surjective proper morphism $f : \mathcal{B}' \to \mathcal{B}$ such that the pullback of $\mathcal{C}^*$ extends to a stable family over $\mathcal{B}'$;
- a complete family of stable curves (over a normal connected base) is uniquely determined by its restriction to a Zariski open subset of the base.

Most importantly, the theory of stable curves exists for fields of any characteristic. Moreover, in the case of finite characteristic p, all morphisms involved are separable (no p-th roots are required).

De Jong's main observation was that this powerful theory provides a tool for desingularization. First of all, every variety can be fibered in curves. Secondly, using induction on dimension and base change we can assume that the base is smooth and that the locus of nonsmooth stable curves is a divisor with normal crossing. The resulting families of stable pointed curves have relatively simple singularities which can be treated directly. One refers to this resolution of singularities as resolution by "alterations". Alterations are not birational but only generically finite!

As it happens, Hironaka's resolution of singularities is, in a sense, too precise. De Jong's alterations are sufficient for many theoretical applications (see below).

2. Outline of the book

Roughly speaking, one can divide the articles of the book into the following categories:

1. method of alterations and its applications in algebraic geometry;
2. applications of desingularization to differential equations;
3. valuation theory;
4. some aspects of the Zariski-Hironaka desingularization theory of algebraic varieties;
5. historical accounts.

The introductory lecture of J. Lipman sketches the biography of Zariski and his main contributions in mathematics. Zariski was a witness and active participant of the turbulent years of the Russian Revolution and Civil War between 1917 and 1920. In 1920 he escaped to Italy and several years later he emigrated to the United States.

The subsequent articles are devoted to different aspects of the desingularization problem in algebraic geometry and some other areas. The first half of the book corresponds roughly to lectures given at the Working Week (unfortunately, not all of them were included). It provides a vigorous introduction to basic problems, classical techniques, and recent developments in the field.

The lecture of H. Hauser gives a concise historical account of the desingularization problem, a list of principal contributions to its solution, selected references, and, most importantly, a convenient dictionary of basic specific terminology used in this book.

The lectures of D. Abramovich and F. Oort explain in a transparent and rigorous manner the proof of the alteration theorem of J. de Jong. They include a thorough exposition of the theory of moduli of stable pointed curves and some results related to de Jong's theorem, in particular several different proofs of the weak version of Hironaka's theorem in characteristic zero.

They are followed by two lectures by J.-M. Aroca devoted to singularities of differential equations and their resolutions by blow-ups. In the first article he gives a detailed proof of the Seidenberg theorem which says that after finitely many blow-ups one can transform a complex foliation of dimension one on a surface to a foliation with simple singularities. Simple means that the first order part is a matrix with at least one nontrivial eigenvalue. This implies that through every point of a foliated surface there passes a locally holomorphic integral curve. Thus, in some cases, blow-ups suffice to reduce these equations, at least locally, to canonical forms. There are examples (Darboux, Jouanolou) of a codimension one foliation in a threefold without (even formal) integral divisors through a singular point.

The lecture notes by S. Encinas and O. Villamayor on constructive desingularization detail recent improvements of Hironaka's approach, giving algorithms for the desingularization of an embedded variety.

The article by G. Bodnár and J. Schicho provides a computer algorithm for the desingularization of an affine hypersurface.

The article of V. Gossard contains a refined version of the desingularization of surfaces, following Zariski's program.

The lecture notes of D. Cox give an introduction to toric varieties, their singularities and toric resolutions. Toric varieties are irreducible algebraic varieties equipped with an action of an algebraic torus (product of several copies of $\mathbf{C}^*$) with a finite number of orbits. Geometric properties of toric varieties (including their singularities) admit a purely combinatorial description. This class of varieties is an ideal testing ground for conjectures: it is sufficiently rich to capture many interesting geometric phenomena and at the same time sufficiently rigid to allow explicit constructions.

The article of B. van Geemen and F. Oort considers compactification of the moduli scheme of curves with non-trivial level structures. They show that there exist natural (though singular) compactifications which are not moduli spaces.

The article of T. Geisser discusses several applications of de Jong's theorem. The first is the non-negativity of local intersection multiplicities in the case of mixed characteristic. Their algebraic definition, in an abstract setting, was given by J.-P. Serre as an Euler characteristic of some explicit derived functor. As intersection multiplicities they ought to be non-negative, but this was not at all clear from the definition. The second application is a theorem about singular cohomology of algebraic varieties (defined by A. Suslin and V. Voevodski). De Jong's theorem implies that singular cohomology (with finite coefficients) for any separated scheme over an algebraically closed field (of characteristic prime to the order of the coefficient ring) coincides with étale cohomology. Further, one obtains a description of Chow groups with finite coefficients as étale cohomology. There are several other applications to relations between different cohomologies and monodromy representations.

R. Goldin and B. Tessier describe a simple toric resolution of a plane curve singularity.

The second article of H. Hauser carefully explains the geometric picture of the resolution of an embedded singular surface.

The short paper by de Jong gives an application of his alteration theorem to Dieudonné modules.

The paper by F.-V. Kuhlmann deals with valuation theory and its connections with logic (model theory for fields).

L. D. Tráng considers M. Spivakovski's approach to surface singularities.

The paper by J. Lipman studies the classical question of a simultaneous resolution of equisingular points. The notion of equisingularity was introduced by Zariski in order to express the intuitive idea that the singular locus admits a natural stratification by algebraic subsets of points with similar complexity of their neighborhoods. Lipman discusses several approaches to the proper definition of this notion.

G. Müller describes resolutions of weighted homogeneous singularities.

F. Pop gives a survey of birational abelian geometry and applications of alterations to the problem of reconstruction of function fields from their Galois groups.

The paper of H. Reitberger recalls several failed attempts to prove resolution of singularities before Hironaka.

The concluding article of M. Vaquié contains the classical treatment of the valuation theory.

3. Comments

Let us make an informal comment on the idea of alterations. Morally, it is the search for *good* covering varieties. Phrased in this way, the theory of alterations hints at the existence of a relatively small class of algebraic varieties which dominate *all* other algebraic varieties. A prototype in arithmetic is a consequence of a theorem by (recently deceased) Russian mathematician G. Belyi, which was noticed by Yu. Manin: for *any* algebraic curve C defined over a number field there exists a modular curve $X_0(N)$ and an unramified covering $X \to X_0(N)$ such that X dominates C (in fact, there are even smaller families of curves with this property). A part of Belyi's argument (and de Jong's lecture at the conference at Santa Cruz) inspired the approach of the first author and T. Pantev to desingularizations. Precisely, any projective algebraic variety is (after blow-ups of some smooth points) a finite covering of a $\mathbb{P}^1$-fibration over a projective space, ramified only in sections (!) of this fibration. In particular, any isolated singularity is a covering of a neighborhood of a smooth point ramified in a family of smooth hypersurfaces. This example indicates that the method of alterations has some hidden potential which remains to be explored.

4. Conclusion

As one can see from our description, many of the articles of the book are either expositions of classical results or useful variations of well known topics. In our opinion, the core of the book consists of the papers discussing different aspects of de Jong's alteration theory and its applications, especially the excellent lecture notes of D. Abramovich and F. Oort. Unfortunately, the presentation of applications of alterations is relatively short. We would have welcomed some comments on the central role of certain classes of singularities and their explicit desingularizations in the minimal model program pursued by Sh. Mori, J. Kollár, and many others (see [4]).

A curious reader will certainly enjoy the multifaceted view of the subject - the emerging internal diversity of the field may provide inspiration to a wide range of mathematicians, from graduate students to experts.

5. Postscriptum

In the autumn of 1981, the first author visited Zariski at his residence with a message from the Harvard Mathematics Department that Zariski was awarded the Wolf Prize. "Too late!" exclaimed the 82-year old mathematician.

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