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A course in p-adic analysis, by Alain M. Robert, Graduate Texts in Math., vol. 198, Springer-Verlag, New York, 2000, xv+437 pp., $54.95, ISBN 0-387-98669-3

The p-adic numbers were discovered by K. Hensel around the end of the nineteenth century. In the course of one hundred years, the theory of p-adic numbers has penetrated into several areas of mathematics, including number theory, algebraic geometry, algebraic topology and analysis (and rather recently to physics, especially in the area of high energy physics via Calabi–Yau manifolds and mirror symmetry).

In the last decades, at least two books have appeared dealing with p-adic numbers and p-adic analysis: F.Q. Gouvêa : *p-adic Numbers*, and A. Escassut: *Analytic elements in p-adic analysis*. The former gives an introduction to p-adic numbers and is elementary. In fact, the book can be (and has been) used for self-study courses by upper-level undergraduates or beginning graduate students in mathematics. The latter is more directed to researchers working in the field of p-adic analysis and mostly concerned with p-adic "analytic" functions.

Robert's book has inevitably some overlaps with the aforementioned two books. Some of the topics in Robert's book that are not discussed in any other p-adic analysis textbooks are topological models of p-adic spaces in Euclidean spaces, a construction of "spherically" complete fields, a p-adic mean value theorem (and its consequences), as well as analytic elements, and some arithmetic applications. Most importantly, Robert's book pays special attention to analytical topics, and indeed, Chapter 6 contains detailed (but friendly) treatments of analytic elements. Chapter 7 gives thorough treatises on arithmetic applications of p-adic analysis. It is indeed these two chapters that give Robert's book somewhat different flavors in comparison with existing textbooks on the subject.

The book consists of seven chapters. Chapter I: p-adic Numbers; Chapter II: Finite Extensions of the Field of p-adic Numbers; Chapter III: Construction of Universal p-adic Fields; Chapter IV: Continuous Functions on $\mathbb{Z}_p$; Chapter V: Differentiation; Chapter VI: Analytic Functions and Elements, and Chapter VII: Special Functions, Congruences. Each chapter contains exercises at the end. Chapters 1, 2 and 3 are supplemented by appendices.

The first five chapters cover standard materials of the subject introducing the reader to constructions of p-adic numbers, p-adic rings, p-adic fields (and their extensions), universal p-adic fields and their basic properties, as well as continuous functions on p-adic rings and their differentiability. The last two chapters are more advanced. In Chapter 6, formal power series are systematically studied aiming toward definition of analytic functions and analytic elements. The last chapter (Chapter 7) discusses several applications of p-adic analysis to arithmetic. For instance, p-adic Gamma function and its properties, a Hazewinkel functional equation lemma and its applications to some arithmetic congruences are the topics treated there.

Now a brief description of each chapter might be in order.

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The first three chapters are devoted to the constructions of p-adic spaces. In Chapter I, the ring $\mathbb{Z}_p$ of p-adic integers and the field $\mathbb{Q}_p$ of p-adic numbers are constructed. These objects are topological with respect to topology induced by the p-adic absolute norm. Their properties as topological groups, rings and fields are discussed. A specially interesting feature is the construction of topological models of $\mathbb{Z}_p$ inside Euclidean spaces. The p-adic field $\mathbb{Q}_p$ is not algebraically closed; that is, it admits algebraic extensions of arbitrary large degrees. This property leads to the main subject of Chapter II, that is, finite extensions of $\mathbb{Q}_p$. Each of these extensions is a finite-dimensional vector space over $\mathbb{Q}_p$. The main result proved in Chapter II is that the p-adic absolute value on $\mathbb{Q}_p$ has a *unique extension* to any finite algebraic extension K of $\mathbb{Q}_p$. To this end, finite-dimensional ultrametric normed spaces (fields), locally compact vector spaces over $\mathbb{Q}_p$, are used as tools. Structures of p-adic fields are studied, including definitions and properties of degree, residue degree, ramification index, and a totally ramified, respectively unramified, field extension. The next natural object is the algebraic closure $\mathbb{Q}_p^a$ of $\mathbb{Q}_p$. However, $\mathbb{Q}_p^a$ is not complete with respect to the underlying topology, so one needs to take its completion, $\mathbb{C}_p$. Unfortunately, this field $\mathbb{C}_p$ is still not large enough to develop p-adic analogues of classical results such as the Hahn–Banach Theorem. This necessitates the construction of a larger field, a so-called *universal p-adic field*, denoted by Ω_p. Such a universal field is a “spherically” complete ultrametric field (that is, all decreasing sequences of closed balls have a nonempty intersection). It is algebraically closed and contains both $\mathbb{Q}_p^a$ and $\mathbb{C}_p$. This is done in Chapter III, first constructing the big ultrametric extension Ω_p using an ultraproduct, and then proving all its necessary properties, and finally defining $\mathbb{C}_p$ as the topological closure of $\mathbb{Q}_p^a$ in Ω_p. This way of constructing a universal p-adic field Ω_p is due to B. Diarra and is one of the distinctive features of this book. With this method the property that $\mathbb{C}_p$ is algebraically closed can be proved rather easily (contrary to a rather lengthy proof with a standard approach).

The next two chapters, Chapters IV and V, study continuous functions defined on subsets of the p-adic field $\mathbb{Q}_p$ with values in an extension of $\mathbb{Q}_p$. In Chapter IV, continuous functions $\mathbb{Z}_p \to \mathbb{C}_p$ are discussed, and some of the results proved here include a p-adic analogue of the Weierstrass Approximation Theorem that any real- or complex-valued function on an interval can be uniformly approximated by polynomial functions; however, there is no canonical series representation for them. The situation is different in the p-adic setting. In fact, special features of p-adic analysis can be summarized in the following theorems.

Theorem. *(a) For any continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, there is a sequence of polynomials $f_n \in \mathbb{C}_p[x]$ that converges uniformly to f.* (Mahler)

(b) There is a canonical Mahler series representation for f. (van der Put)

These theorems are proved analyzing locally constant functions on $\mathbb{Z}_p$. Then ultrametric Banach spaces (i.e., complete infinite-dimensional ultrametric normed spaces) are studied culminating with the p-adic Hahn–Banach theorem. Finally a generalization of the Mahler theorem due to L. van Hamme is proved. The proof makes use of umbral calculus, and a brief review of this topic is presented. In Chapter V, calculus in the p-adic domain is the main topic of discussion. The topics discussed include differentiability, a p-adic mean value theorem (and its consequences), differentiability of Mahler series, and the Volkenborn integral. A presentation of a p-adic mean value theorem (with small enough increments) is one of the distinctive features of this book. Furthermore, the exponential function and logarithm are studied in detail, including: definitions, basic properties, derivatives, continuation of the logarithm and extension of the exponential to the whole of $\mathbb{C}_p$. Also relation of the Volkenborn integral to Bernoulli numbers and polynomials and its application to sums of powers are discussed, culminating with the Clausen-von Staudt theorem.

The last two chapters (Chapters 6 and 7) contain more advanced topics. Chapter 6 discusses analytic functions and analytic elements by way of a thorough and systematic study of formal power series. The exponential and the logarithm functions serve as the guiding examples for this endeavor (as well as, of course, theorems in complex analysis). Formal power series, formal derivations (and Chain Rule), convergent power series (and radii of convergence), formal substitutions, Newton polygons and their dual polygons (valuation polygons), and Laurent series are rigorously defined, and their basic properties are discussed. Then zeros of restricted power series, e.g., finiteness of zeros (a theorem of Strassman), and existence of zeros are proved. Entire functions in p-adic analysis can be completely classified:

Theorem. *Let $f(x)$ be a formal power series over a complete extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ with infinite radius of convergence. Then either f is a nonzero constant (if it does vanish), a polynomial (if it has finitely many zeros), or a convergent infinite product (if it has infinitely many zeros).*

Further, a p-adic version of Rolle's theorem, the maximum principle, and extensions to Laurent series (Hadamard's Three-Circle Theorem, Theta Functions, etc.), are discussed. Functions defined by convergent power series are defined in a ball, and they are not amenable to analytic extensions, as there is no way of defining "points near the boundary" of a ball. This leads to a new way of defining analytic functions. This is done by considering rational functions (quotients of polynomial functions). Rational functions are expanded to power series in each ball containing none of their poles. Their basic properties, e.g., Mittag–Leffler decompositions, product decompositions (Motzkin decompositions), are discussed. Then thorough yet elementary treatments of analytic elements are presented, and this is one of the highlights of this book. Analytic elements were introduced by Krasner in p-adic analysis to mimic Runge's theorem in complex analysis that a holomorphic function defined in a domain $D \subset \mathbb{C}$ can be uniquely approximated by rational functions. Analytic elements are defined as uniform limits of sequences of rational functions and provide tools for analytic continuations in p-adic analysis. Then several characterization theorems for analytic elements are proved. Let $\mathbf{M}_p$ denote the open balls $\{|x| < 1\}$, $\mathbf{A}_p = \{|x| \leq 1\}$, $B = 1 + \mathbf{M}_p$ the open ball of radius 1 with center 1, and B^c the complement of B. Let $H_0(B^c)$ denote the space of formal power series of the form $f = \sum_{n \geq 1} a_n/(x-1)^n$ with $|a_n|/r^n \to 0 (n \to \infty)$. Then characterization theorems are formulated as follows:

Theorem. *(a) The space $H(\mathbf{A}_p)$ of analytic elements on the closed unit ball $\mathbf{A}_p$ coincides with the Tate algebra $\mathbb{C}_p\{x\}$ with norm $||f|| = sup_{|x| \leq 1}|f(x)|$.*

(b) Let $f = \sum_{n \geq 0} a_n x^n \in \mathbb{C}_p[[x]]$ be a convergent power series with radius of convergence ≥ 1. Then the sequence $n \mapsto a_n$ has a continuous extension $\phi : \mathbb{Z}_p \to \mathbb{C}_p$ if and only if f is the restriction of an analytic element of $H_0(B^c)$ (Amice and Fresnel).

(c) Let $f = \sum_{n \geq 0} a_n x^n \in \mathbb{C}_p[[x]]$ *be a formal power series with bounded coefficients. Define* $p_\nu = p^\nu(p^\nu - 1)\,(\nu \geq 1)$. *Then* f *defines an analytic element on* $\mathbf{M}_p$ *precisely when the following condition (CR) holds:*

For each $\epsilon > 0$ *there exist* ν *and* $N \geq 0$ *such that* $|a_{n+p^\nu} - a_n| \leq \epsilon$ $(n \geq N)$. (Christol and Robba).

(d) Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ *be a continuous function with Mahler series* $f(x) = \sum_{k \geq 0} c_k \binom{x}{k}$. *Then* f *is the restriction of an analytic element* $\tilde{f} \in \mathbb{C}_p\{x\}$ *if and only if* $|c_k/k!| \to 0$.

Finally Chapter 7 is devoted to the study of special functions and applications of p-adic analysis to arithmetic congruences. This chapter aims to give unified treatments by p-adic means of many special functions and arithmetic congruences. Specifically, it contains detailed discussions of the Morita p-adic Gamma function, the Artin–Hasse exponential function, the Honda congruences, and their applications. More importantly, it provides a glimpse to vast arithmetic (and possibly physical) applications of p-adic analysis (yet to be explored to its fullest extent). Let $p \geq 3$ be a prime. The Morita p-adic Gamma function is the continuous function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$ that extends the function

$$f(n) := \prod_{1 \leq j < n,\, p \nmid j} j \quad (n \geq 2).$$

Basic properties of Γ_p (some of which are analogous to the classical Gamma function) are studied, e.g., the p-adic Gauss multiplication formula, a Mahler series representation of Γ_p, and the power series expansion of $\log \Gamma_p$. Then a theorem of Kazandzidis on congruences for some binomial coefficients modulo higher powers of p is proved:

Theorem (Kazandzidis)**.** *For all primes* $p \geq 5$, *one has*

$$\binom{pn}{pk} \equiv \binom{n}{k} \quad (\mathrm{mod}\ p^3 nk(n-k) \binom{n}{k} \mathbb{Z}_p).$$

For $p = 3$, *the same congruence holds modulo* $3^2 nk(n-k) \binom{n}{k} \mathbb{Z}_3$ *(namely one power of p fewer).*

The Kazandzidis congruence is proved using the p-adic Gamma function. In fact, it follows from the following result:

Theorem. *Let* $f(x) = log\Gamma_p(px)$ $(x \in \mathbb{Z}_p)$. *Then*

(a) f *is given by a restricted series having all its coefficients in* $p\mathbb{Z}_p$,

(b) $|f(x+y) - f(x) - f(y)| \leq |p^3 xy(x+y)|$.

To complete the discussion on p-adic Gamma functions, the 2-adic Gamma function Γ_2 is also defined.

The Artin-Hasse exponential function is the formal power series defined by

$$E_p(x) := exp(x + \frac{1}{p} x^p + \frac{1}{p^2} x^{p^2} + \cdots) = 1 + x + \cdots .$$

Its basic properties are studied. In particular, the p-integrality of $E_p(x)$ is proved expressing $E_p(x)$ in terms of the binomial series $(1 - x^n)^{-\mu(n)/n}$ where μ is the Möbius function. Another way of establishing the p-integrality of $E_p(x)$ is to invoke the Dieudonné–Dwork criterion. The principle of the Dieudonné–Dwork criterion is that the two operations:

(a) *first raising x to its p-th power x^p and then computing $f(x^p)$*, and

(b) *first computing $f(x)$ and then raising it to the p-th power $f(x)^p$*,

give rise to similar results. A more precise formulation is given in the following theorem.

Theorem (Dieudonné–Dwork)**.** *Let $f(x) \in 1 + x\mathbb{Q}_p[[x]]$ be a formal power series. Then f has p-integral coefficients, i.e., in $\mathbb{Z}_p$, if and only if*

$$f(x)^p / f(x^p) \in 1 + px\mathbb{Z}_p[[x]].$$

Another topic discussed here is the construction due to Dwork of a p-th root of unity in $\mathbb{C}_p$. For this, Dwork used the so-called Dwork series:

$$E_\pi(x) := exp(\pi(x - x^p)) \in \mathbb{Q}_p(\pi)[[x]]$$

(with π a root of $x^{p-1} + p = 0$ in $\mathbb{Q}_p^a$).

Theorem. *Let π be a root of $x^{p-1} + p = 0$. Then $K = \mathbb{Q}_p(\pi)$ is a Galois extension of $\mathbb{Q}_p$, which is a totally ramified extension of degree $p - 1$ and $K = \mathbb{Q}_p(\mu_p)$. More precisely:*

(a) The field K contains a unique p-th root of unity $\zeta_\pi \in \mu_p$ such that

$$\zeta_\pi \equiv 1 + \pi \pmod{\pi^2}.$$

(b) The Dwork series $E_\pi(x)$ has a radius of convergence $p^{(p-1)/p^2} > 1$.

(c) For every $a \in \mathbb{Q}_p$ with $a^p = a$, one has

$$E_\pi(a) \in \mu_p, \quad E_\pi(a) \equiv 1 + a\pi \pmod{\pi^2}$$

so that $E_\pi(1) = \zeta_\pi$.

The existence of roots of unity in $\mathbb{C}_p$ naturally leads to Gauss sums. Let $p \geq 3$ be a prime, and let $\mathbb{F}_q$ (with $q := p^f$) be the finite field with q elements. Let $\psi : \mathbb{F}_q \to K^\times$ and $\chi : \mathbb{F}_q^\times \to K^\times$ be additive and multiplicative characters of $\mathbb{F}_q$, respectively. The Gauss sum attached to a pair (ψ, χ) is the sum

$$G(\psi, \chi) := \sum_{v \in \mathbb{F}_q^\times} \psi(v)\chi(v).$$

Specially interesting Gauss sums (for $\mathbb{F}_p$) are of the form:

$$G_\alpha = -\sum_{0 \neq x \in \mathbb{F}_p} \omega(x)^{-(p-1)\alpha} \zeta_\pi^x$$

where $\omega(x) \in \mu_{p-1}$ is the unique root of unity in $K = \mathbb{Q}_p(\mu_p)$ having reduction x in the residue field. The Gross–Koblitz formula for G_α gives a relation between G_α and the p-adic Gamma function. More precisely, write $\alpha = \frac{a}{p-1}$ with $0 \leq a < p - 1$. Then

$$G_\alpha = \pi^a \Gamma_p(\frac{a}{p-1}).$$

Actually, a more general Gross–Koblitz formula is proved for Gauss sums (for $\mathbb{F}_q$). Let $K = \mathbb{Q}_p(\pi, \mu_{q-1}) \subset \mathbb{C}_p$ be the tamely ramified extension with ramification index

$p-1$. Let $\alpha \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$. Choose a representation of α and write $0 \le \alpha = \frac{a}{q-1} < 1$. Then expand a p-adically as

$$a = a_0 + a_1 p + \cdots + a_{f-1}p^{f-1} < q-1 < q.$$

Let $S_p(a) := \sum_{0\le j<f} a_j$. Define the integers $a^{(i)} (1 \le i \le f-1)$ by the i-th cyclic permutation of a, that is,

$$\begin{aligned} a^{(1)} &= a_{f-1} + a_0 p + \cdots + a_{f-2}p^{f-1}, \\ a^{(2)} &= a_{f-2} + a_{f-1}p + \cdots + a_{f-3}p^{f-1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a^{(f-1)} &= a_1 + a_2 p + \cdots + a_0 p^{f-1}. \end{aligned}$$

Let $Tr : \mathbb{F}_q \to \mathbb{F}_p$ be the trace:

$$Tr(x) = x + x^p + \cdots + x^{p^{f-1}}.$$

Theorem (Gross–Koblitz)**.** *Let* $0 \le \alpha = \frac{a}{q-1} < 1$. *Then the value of the Gauss sum* G_α *is given by*

$$G_\alpha = -\sum_{0\ne x\in\mathbb{F}_q} \omega(x)^{-a}\psi(Tr(x)) = \pi^{S_p(a)} \prod_{0\le j<f} \Gamma_p(\frac{a^j}{q-1}).$$

Finally the Honda congruences are studied in detail. Suppose that the rings A and B are either $A = \mathbb{Z}_{(p)}[t] \subset B = \mathbb{Q}[t]$, or $A = \mathbb{Z}_p[t] \subset B = \mathbb{Q}_p[t]$. Let $\sigma : B \to B$ be the $\mathbb{Q}$ (resp. $\mathbb{Q}_p$)-linear map sending $a(t) \to a(t^p)$, which is extended to

$$\sigma_* : B[[x]] \to B[[x]] \quad ; \quad \sum_{i\ge 0} a_i(t)x^i \mapsto \sum_{i\ge 0} a_i(t^p)x^i.$$

We call any map $H_p : B[[x]] \to B[[x]]$ of the form

$$f \mapsto H_p f := f(x) - \frac{1}{p}\sum_I \sigma_*^i f(x^{p^i})$$

(with $I \subset \{1,2,3,\cdots\}$) a *Hazewinkel* map.

A sequence $(a_m)_{m\ge 1}$ in $A = \mathbb{Z}_{(p)}[t]$ is called a *p-Honda sequence* if it satisfies the congruence

$$a_m(t) \equiv a_{m/p}(t^p) \pmod{m\mathbb{Z}_{(p)}[t]} \quad \text{when } p|m.$$

In particular, a sequence $(a_m)_{m\ge 1}$ in $\mathbb{Z}_p$ is a *p-Honda sequence* if $a_m \equiv a_{m/p}$ (mod $m\mathbb{Z}_{(p)}$) when $p|m$.

Characterizations of p-Honda congruences are obtained:

Theorem. *For a formal power series* $f(x) = \sum_{m\ge 1} a_m x^m/m \in B[[x]]$, *the following statements are equivalent:*

(i) $H_p f = f(x) - \frac{1}{p}\sigma_* f(x^p)$ *has its coefficients in* $A \subset B$,

(ii) $exp(f(x))$ *is p-integral, i.e., has coefficients in A.*

Several examples of p-Honda congruences are discussed.

Applications. *(a) Let M be a square integer matrix of order d. Then the sequence* $(Tr(M^m))_{m\ge 1}$ *is a p-Honda congruence for any prime p. That is,*

$$Tr(M^m) \equiv Tr(M^{m/p}) \pmod{m\mathbb{Z}_p} \quad \text{if } p|m$$

(Beukers).

(b) The Legendre polynomial $P_m(t)(m \geq 0)$ *is defined by*

$$P_m(t) = \frac{1}{2^m m!}(\frac{d}{dt})^m (t^2 - 1)^m.$$

Then for any odd prime $p|m$, *the Legendre polynomials satisfy the following two Honda congruences:*

$$P_{m-1}(t) \equiv P_{(m/p-1)}(t^p) \pmod{m\mathbb{Z}_{(p)}[t]}$$

and

$$P_m(t) \equiv P_{m/p}(t^p) \pmod{m\mathbb{Z}_{(p)}[t]}.$$

(c) Let $(A_m(t))_{m\geq 0}$ *be an Appell systems of polynomials in* $\mathbb{Z}_p[t]$*; that is,*

$$\deg A_m = m, \quad A'_m = mA_{m-1}, \quad (m \geq 1).$$

Then $(A_m(t))_{m\geq 1}$ *is a Honda sequence of polynomials; i.e.,*

$$A_m(t) \equiv A_{m/p}(t^p) \pmod{m\mathbb{Z}_p[t]}, \quad (m \geq 1, p|m),$$

if and only if there exists $a \in \mathbb{Z}_p$ *such that* $(A_m(a))_{m\geq 1}$ *is a Honda sequence, if and only if there exists* $a \in \mathbb{Z}_p$ *such that*

$$A_m(a) \equiv A_{m/p}(a^p) \pmod{m\mathbb{Z}_p}, \quad (m \geq 1, p|m)$$

(Zuber).

The book is a pleasure to read, and I would recommend it for anyone interested in learning p-adic analysis from scratch. Especially, the expositions are so friendly that graduate students, mathematicians and physicists (with some knowledge in complex analysis) should be able to read it through without much difficulty. This book may also be used as a textbook for seminar-type courses directed to upper-level undergraduates or beginning graduate students. The book is self-contained, and in fact, some necessary materials not in the main texts are supplied in appendices. Exercises at the end of each chapter provide supplements to the main text. The reader is encouraged to do these exercises.

To the reviewer, the last chapter is the most interesting, and she would have liked to have more expositions of arithmetic applications. However, applications to number theory (Iwasawa theory), arithmetic algebraic geometry (p-divisible groups, formal Brauer groups, etc.) as well as mathematical physics (Calabi–Yau manifolds, mirror symmetry) would be the topics of another book!

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