BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, Pages 129–135
S 0273-0979(01)00928-4
Article electronically published on October 12, 2001

Enumerative combinatorics, Volume 2, by Richard P. Stanley, Cambridge University Press, Cambridge, 1999, xii+581 pp., \$74.95, ISBN 0-521-56069-1

The fundamental problem of enumerative combinatorics is to determine the number of elements of a set. More precisely, given an infinite indexed collection $\{A_i\}_{i\in I}$ of finite sets, we want to find a formula for the cardinality of A_i as a function of i, or at least a method for determining the number of elements of A_i that is easier (or more interesting) than counting them one at a time. There are many interesting mathematical problems that can be phrased as counting problems but to which the methods of enumerative combinatorics do not apply; for example, problems of counting groups will generally use the techniques of group theory rather than of enumerative combinatorics, and some seemingly reasonable problems, such as counting transitive relations on a set of n elements, turn out not to be amenable to the methods of enumerative combinatorics. But surprisingly many different kinds of objects can be counted, and counting them often involves interesting mathematics.

The two fundamental tools of enumerative combinatorics are *bijections* and *generating functions*. Two sets A and B have the same cardinality if and only if there is a bijection from A to B, so a bijection from A to B allows us to count the elements of A by counting the elements of B. As a very simple example, a *composition* of an integer m is a sequence $(a_1, a_2, \ldots, a_k)$ of positive integers with sum m. There is a bijection from the set of compositions of $n + 1$ to the set of subsets of $\{1, 2, \ldots, n\}$: the composition $(a_1, \ldots, a_k)$ corresponds to the subset $\{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{k-1}\}$. So from the fact that there are 2^n subsets of $\{1, 2 \ldots, n\}$, we get that there are 2^n compositions of $n + 1$.

A more interesting example is given by the enumeration of *Dyck words*, which are words in the letters X and Y with as many X's as Y's, and with the property that any initial segment contains at least as many X's as Y's. For example, $XXYXYY$ is a Dyck word, but $XXYXYYYX$ is not. If we replace each X with a left parenthesis and each Y with a right parenthesis, then a Dyck word becomes a sequence of paired parentheses; thus $XXYXYY$ becomes $(()())$. We can count Dyck words of length $2n$ by starting with the $\binom{2n}{n}$ words made up of n X's and n Y's and subtracting the number of these words that are not Dyck words. A bijection due to D. André [2] shows that $\binom{2n}{n-1}$ of these words are not Dyck words: Given a word with n X's and n Y's that is not a Dyck word, find the first Y that violates the condition and interchange all X's and Y's that occur after this Y. (For example, $XXYYYXXY$ becomes $XXYYYYYX$.) We obtain a word with $n - 1$ X's and $n + 1$ Y's; this gives a bijection from the set of words with n X's and n Y's that are not Dyck words to the set of all words with $n - 1$ X's and $n + 1$ Y's. Thus the number of Dyck words of length $2n$ is the *Catalan number* $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.

2000 *Mathematics Subject Classification.* Primary 05A15, 05E05; Secondary 05A05, 05A10, 05E10.

Recurrences can be obtained from bijections. Every nonempty Dyck word d has a unique factorization

$$d = d_1 X d_2 Y, \tag{1}$$

where d_1 and d_2 are Dyck words, and this factorization gives a bijection from Dyck words of length $n > 0$ to ordered pairs of Dyck words of total length $n-1$, which yields a recurrence for the number C_n of Dyck words of length n:

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}. \tag{2}$$

The *generating function* for a sequence $a_0, a_1, \dots$ is the formal power series $a_0 + a_1 x + a_2 x^2 + \cdots$. Generating functions are often presented as a method for solving recurrences. Thus the recurrence for the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$, yields the equation $f(x) = 1 + (x + x^2)f(x)$ for the generating function $f(x) = F_0 + F_1 x + \cdots$, which gives $f(x) = x/(1 - x - x^2)$, and the recurrence (2) for the Catalan numbers yields the equation

$$C(x) = 1 + xC(x)^2 \tag{3}$$

for the generating function $C(x) = C_0 + C_1 x + \cdots$; solving for $C(x)$ gives

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

However, a different, and more useful, interpretation of generating functions explains formulas like (3) with no need to write down a recurrence like (2). To "count" a set S, not necessarily finite, we assign to each element u of S a *weight* $w(u)$ in some formal power series ring. Then the generating function of S with respect to the weight function w is $\sum_{u \in S} w(u)$ (where we require that the sum exist). A decomposition of S that is compatible with w gives us an identity involving the generating function of S. For example, we may assign a Dyck word of length $2n$ the weight x^n. Then the factorization (1) yields $w(d) = xw(d_1)w(d_2)$; so if we define $C(x)$ to be the sum of the weights of all Dyck words, we obtain (3) directly from (1).

Richard Stanley's *Enumerative Combinatorics*, Volume 1 [13], appeared in 1986, and its interesting choice of topics, clear prose style, numerous exercises, and scholarly erudition quickly won it a large following. The long-awaited Volume 2 appeared in 1999, and it is an impressive work of scholarship that surpasses the high standards set by Volume 1. Stanley was the awarded the American Mathematical Society's 2001 Leroy P. Steele Prize for Mathematical Exposition in recognition of the completion of the two volumes.

Volume 2 continues the chapter numbering of Volume 1, so its three chapters are numbered 5, 6, and 7. In Chapter 5, Stanley covers two of the most important topics in enumerative combinatorics not covered in Volume 1: exponential generating functions and Lagrange inversion. Suppose that for any set S of n "labels", we can make a_n different structures of some type out of these labels. Then we define the *exponential generating function* for this type of structure to be $\sum_n a_n x^n/n!$. For example, there are $n!$ permutations of a set of n labels, so the exponential generating function for permutations is $\sum_n n!\, x^n/n! = (1-x)^{-1}$; there are $2^{\binom{n}{2}}$ graphs with a given n-element vertex set, so the exponential generating function for graphs is $\sum_n 2^{\binom{n}{2}} x^n/n!$. Exponential generating functions are useful in counting labeled structures because of the way they multiply. If $a(x) = \sum_n a_n x^n/n!$ and $b(x) = \sum_n b_n x^n/n!$, then $a(x)b(x) = \sum_n c_n x^n/n!$, where

$$c_n = \sum_k \binom{n}{k} a_k b_{n-k}.$$

To interpret this formula combinatorially, suppose that α and β are types of structures with exponential generating functions $a(x)$ and $b(x)$. Then $c(x)$ is the exponential generating function for structures consisting of an ordered pair of two structures, the first of type α and the second of type β: Given a set of n labels, we construct such a pair by choosing an integer k, choosing a k-element subset of the label set in $\binom{n}{k}$ ways, and then choosing a structure of the first type on these k labels in a_k ways and choosing a structure of the second type on the remaining $n-k$ labels in b_{n-k} ways. (A more formal account of labeled structures can be found in [3].) For example, every permutation can be represented as a pair consisting of a derangement (permutation without fixed points) and a set of fixed points, so if $D(x)$ is the exponential generating function for derangements, then $(1-x)^{-1} = D(x)e^x$, and thus $D(x) = e^{-x}/(1-x)$.

In addition to multiplying exponential generating functions, we can also compose them. Since $a(x)^k$ counts k-tuples of structures of the type counted by $a(x)$, if $a(x)$ has no constant term, we can divide by $k!$ to see that $a(x)/k!$ counts k-element sets of structures, each counted by $a(x)$, and thus $e^{a(x)}$ counts sets of an arbitrary number of these structures. This "exponential formula" has many applications. For example, since any graph may be viewed as a set of connected graphs, the exponential generating function $c(x)$ for connected graphs satisfies the equation $e^{c(x)} = \sum_n 2^{\binom{n}{2}} x^n/n!$, from which its coefficients can easily be computed.

Trees are connected graphs without cycles. Arthur Cayley [4], in one of the earliest results in enumerative combinatorics, proved (or at least asserted) that there are n^{n-2} trees on n vertices. To prove Cayley's formula using exponential generating function, we choose one of the n vertices of a tree to be the "root", so Cayley's formula is equivalent to the statement that there are n^{n-1} rooted trees on n vertices. A rooted tree can be obtained by taking a set of rooted trees and a new root and adding an edge from every old root to the new root. This construction gives a functional equation for the exponential generating function $R(x)$ for rooted trees:

$$R(x) = xe^{R(x)}. \tag{4}$$

The equation (4) is typical of functional equations that arise in enumerative combinatorics. It can be solved by the *Lagrange inversion formula*, which says that if $f(x) = xG(f(x))$, then the coefficient of x^n in $f(x)^k$ is k/n times the coefficient of x^{n-k} in $G(x)^n$. Stanley gives three proofs of the Lagrange inversion formula: one is an algebraic proof, using properties of formal power series; one uses exponential generating functions and a bijection due to Prüfer [11] that gives a direct proof of a refinement of Cayley's formula; and one uses a different bijective proof of a formula for counting generalizations of Dyck words, with the interpretation of the equation to be solved similar to that of (3).

The rest of Chapter 5 deals with two related, but more specialized, topics: exponential structures, which give a poset-theoretic approach to generalizations of the exponential formula, and the matrix-tree theorem, which gives a determinantal formula for counting spanning trees of a graph.

After the text of of each chapter come historical notes, with complete references. Chapter 5 has four pages of notes and 70 references. The chapter concludes with 74 exercises, most with more than one part. These exercises contain an enormous amount of additional material on topics including threshold graphs, Eulerian polynomials, degree sequences of graphs, Mehler's formula for Hermite polynomials, blocks, polynomials of binomial type, series-parallel posets, alternating trees, noncrossing partitions, and parking functions, and all have either a solution or a reference to the literature.

Chapter 6 covers three classes of formal power series, together with their enumerative applications: algebraic, D-finite, and noncommutative. (Rational power series were discussed in Volume 1.) The chapter begins with a discussion of some of the connections between algebraic functions and formal power series, notably Puiseux's theorem, which implies that any algebraic function of x can be expressed as a Laurent series in some fractional power of x. Next, Stanley considers some algebraic functions that arise in enumeration problems whose coefficients generalize the Catalan numbers. For any set S of positive integers, let $u = u(x,t)$ satisfy

$$u = x\left(t + \sum_{j\in S} u^j\right). \tag{5}$$

Then the coefficients of u have combinatorial interpretations in terms of plane trees, lattice paths, dissections of a polygon, and parenthesizations. The quadratic cases of (5) are especially important and give rise to several much-studied sequences, further properties of which appear in the exercises to this chapter. If $S = \mathbb{P}$ (the set of positive integers), the coefficients are *Narayana numbers*; if $x = 1$ and $S = \mathbb{P} - \{1\}$, they are *Schröder numbers*; if $t = 1$ and $S = \{1, 2\}$, they are *Motzkin numbers*; and if $x = 1$ and $S = \{2\}$ or if $t = 1$ and $S = \mathbb{P}$, they are Catalan numbers.

Although most occurrences of algebraic generating functions in enumeration correspond to equations like (5) whose combinatorial interpretation is straightforward, there are examples where algebraicity is not so obvious. One of the most interesting, described in problem 6.41, involves "2-stack-sortable" permutations. Given any sequence w of distinct integers, factor w as unv, where n is the largest entry of w. Define the "stack-sorting" operator S recursively by $S(w) = S(u)S(v)n$, where S takes the empty sequence to itself. A permutation π of $\{1, 2, \ldots, n\}$ is called *stack-sortable* if $S(\pi) = 1\,2\cdots n$ and *2-stack-sortable* if $S(S(\pi)) = 1\,2\cdots n$. It is easy to show that the number stack-sortable permutations is the Catalan number C_n. It is quite difficult to show (as was conjectured by Julian West and proved by Doron Zeilberger [14]) that the number of 2-stack sortable permutations of $\{1, 2, \ldots, n\}$ is $2(3n)!/(n + 1)!\,(2n + 1)!$. The generating function is algebraic, but not for any obvious combinatorial reason.

D-finite (also called *holonomic*) power series include the better known algebraic power series. A power series $f(x)$ is D-finite if it satisfies a linear homogeneous differential equation with coefficients that are polynomials in x, or equivalently, if the set of its derivatives $d^k f/dx^k$ spans a finite dimensional vector space over the field of rational functions of x. This property implies that the coefficients of f satisfy linear homogeneous recurrence relations with polynomial coefficients and can therefore be computed efficiently.

Noncommutative generating functions arise very naturally in enumeration. The set of words in an alphabet becomes a monoid under the operation of concatenation, and taking formal sums gives an algebra of formal power series in noncommuting variables. One can define rational and algebraic noncommutative power series (though the definitions are more complicated than one might expect), and Stanley gives a nice exposition of their basic properties. As an example of a noncommutative algebraic series, we may consider the sum D of all Dyck words as a formal power series in the noncommuting variables X and Y. The factorization (1) gives the equation $D = 1 + DXDY$, and this implies that D is algebraic.

Chapter 6 contains 74 exercises. A highlight is exercise 6.19, which contains 66 combinatorial interpretations of the Catalan numbers. Many of the other exercises involve the Catalan numbers and their relatives, but some contain results less closely related to combinatorics (and unlikely to be familiar to most combinatorialists). One that I found particularly intriguing is exercise 6.4, in which Stanley points out that Puiseux's theorem fails in positive characteristic, and gives the characteristic p example $y^p - y - x^{-1}$, due to Chevalley [5, §IV.6], whose roots are not fractional Laurent series but which has the factorization

$$y^p - y - x^{-1} = \prod_{i=0}^{p-1}\left(y - i - \sum_{j\geq 1} x^{-1/p^j}\right)$$

due to Abhyankar [1].

Chapter 7, by far the longest of the three chapters, is on symmetric functions. A formal power series in the infinitely many variables $x_1, x_2, \dots,$ is called *symmetric* if it is invariant under any permutation of the variables. These symmetric formal power series are traditionally called *symmetric functions* even though they are not functions in the usual sense. The homogeneous symmetric functions of degree n form a vector space, denoted Λ^n, whose dimension is the number of partitions of n. There are several important bases for Λ^n. If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n, where $\lambda_1 \geq \cdots \geq \lambda_k$, then the *monomial symmetric function* m_λ is the sum of all distinct monomials of the form $x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}$ for some permutation $(\alpha_1, \dots, \alpha_n)$ of λ. For each integer $r \geq 0$, the rth *elementary symmetric function* e_r is the sum of all products of r distinct variables, so $e_0 = 1$, and for $r > 0$,

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The rth *complete symmetric function* h_r is the sum of all monomials of degree r, so $h_0 = 1$, and for $r > 0$,

$$h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The rth *power sum symmetric function*, for $r \geq 1$, is

$$p_r = \sum_i x_i^r.$$

For any partition $\lambda = (\lambda_1, \dots, \lambda_k)$, we define $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$, $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$. Then $\{m_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$, and $\{p_\lambda\}$, where λ runs over all partitions of n, are bases for Λ^n. There is a symmetric scalar product on Λ^n for which the monomial and complete symmetric functions are biorthogonal ($\langle h_\lambda, m_\mu \rangle = \langle m_\mu, h_\lambda \rangle = \delta_{\lambda,\mu}$) and the power sums are orthogonal (but not orthonormal).

There is a fifth important basis for Λ^n which is less obvious and more interesting: the Schur functions. They may be defined most simply as determinants in the complete symmetric functions. If λ is a partition with n parts, we define the *Schur function* s_λ by

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n},$$

where we take $h_m = 0$ for $m < 0$.

The coefficients of the Schur functions have a very interesting combinatorial interpretation, which is the focus of Stanley's approach, in contrast to Macdonald [8], the standard reference on this subject. If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n, a *semi-standard Young tableau* (SSYT) of shape λ is an array of n boxes, with λ_i left-justified boxes in row i, filled with positive integers that are weakly increasing in rows and strictly increasing in columns. For example,

1	1	2	4
2	3		
3	5		

is an SSYT of shape $(4, 2, 2)$. Then the coefficient of $x_1^{r_1} \cdots x_j^{r_j}$ in s_λ is the number of SSYT of shape λ containing r_i entries equal to i for each i. The key to the combinatorial theory of Schur functions is an insertion algorithm, due (in a slightly less general form) to Craige Schensted [12]: given an SSYT P and a positive integer k, Schensted's algorithm creates a new SSYT P' whose entries are those of P together with k, and whose shape is the shape of P with an additional box added. Schensted's insertion algorithm is bijective in the sense that if we know P' and the shape of P, then we can recover P. An important consequence of Schensted's insertion algorithm is the *Robinson-Schensted-Knuth (RSK) algorithm*, which gives a bijection between ordered pairs of SSYT of the same shape and matrices of nonnegative integers. The RSK algorithm yields the *Cauchy identity*

$$\sum_\lambda s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, y_2, \dots) = \prod_{i,j} \frac{1}{1 - x_i y_j},$$

which implies, with a bit of linear algebra, that the Schur functions are orthonormal.

The Schur functions are closely related to the irreducible representation of the symmetric and general linear groups. It follows from the orthonormality of the Schur functions that when they are expanded in power sum symmetric functions, the coefficients are (up to an easily described factor) the values of the irreducible characters of symmetric groups. Moreover, the Schur functions in a finite number of variables, interpreted as functions of the eigenvalues of a matrix, are the characters of the irreducible polynomial representations of general linear groups.

In addition to the basic theory of symmetric functions, Chapter 7 discusses several related topics: quasi-symmetric functions, enumeration of plane partitions, and enumeration under group action (Pólya theory [9], [10]).

There are two appendices to Chapter 7. The first appendix (by Sergey Fomin) discusses some additional combinatorial constructions on tableaux and includes a proof of the *Littlewood-Richardson rule*, which gives a combinatorial interpretation to the coefficients in the expansion of a product of two Schur functions into Schur functions. The second appendix explains the connection between Schur functions and representations of general linear groups.

The two volumes of *Enumerative Combinatorics* cover nearly every major topic in enumerative combinatorics to a greater or lesser degree. The most important topic I can think of that is not mentioned at all is the enumeration of planar maps. (A good account can be found in Goulden and Jackson [6]; they also treat sequences and lattice paths more extensively than does Stanley.) A topic that receives only brief coverage is that of enumeration under group action, and in particular the enumeration of unlabeled graphs of various types. A comprehensive account of the theory up to 1972 was given by Harary and Palmer [7], and a more modern approach to the subject has been given recently by Bergeron, Labelle, and Leroux [3]. However, an exposition in which symmetric functions appear in the central role that (in this reviewer's opinion) they deserve does not yet exist.

Enumerative Combinatorics can be read on two levels. The text, together with the easier exercises, is a very thorough introduction to enumerative combinatorics that is accessible to a beginning graduate student or even a good undergraduate. With its more advanced exercises, historical notes, and extensive references, the books are an indispensable resource for the expert. A supplement, which includes errata, updates, and new material, can be downloaded from the author at `http://www-math.mit.edu/~rstan/ec/`.

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