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Tree lattices, by Hyman Bass and Alexander Lubotzky, with appendices by H. Bass, L. Carbone, A. Lubotzky, G. Rosenberg, and J. Tits, Prog. Math., vol. 176, Birkhäuser Boston, Inc., Boston, MA, 2001, xiv + 233 pp., $54.95, ISBN 0-8176-4120-3

My first acquaintance with the book under review dates back to my graduate school years. At the end of every school year, my classmates and I used to ask our advisor, G. A. Margulis, to recommend a book or a topic to study over the summer in our student seminar. One summer the suggestion was *Tree lattices* by Hyman Bass and Alex Lubotzky, which at that time was only a draft that circulated from person to person. I must say in retrospect that each of my advisor's recommendations turned out to be most useful in my subsequent career and was also great mathematical literature. Needless to say, I was thrilled when the book *Tree lattices* [BL] came out in its present form and rushed to purchase a copy. I was not disappointed: the book contains many new results that appeared only as conjectures in the earlier draft.

This book presents an extensive study of tree lattices, which we will define. Let X be a locally finite tree. Then $\mathcal{G} = AutX$ is a locally compact group. The vertex stabilizers $\mathcal{G}_x$ are open and compact. A subgroup $\Gamma \subset \mathcal{G}$ is *discrete* if Γ_x is finite for some and hence all $x \in VX$. In this case one can define

$$Vol(\Gamma\backslash\backslash X) := \sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|}.$$

We call Γ *a lattice* if $Vol(\Gamma\backslash\backslash X) < \infty$. We say Γ is a *uniform* lattice if $\Gamma\backslash X$ is a finite graph. A classical example of tree lattices is provided by consideration of rank-1 Lie groups defined over a non-archimedean local field. For instance, let $F = \mathbb{F}_q((\frac{1}{t}))$ be a local field of Laurent formal power series over a finite field of cardinality q. Then the group $SL_2(F)$ acts on the associated Bruhat-Tits tree X (see [Se]) and $\Gamma = SL_2(\mathbb{F}_q[t])$ is a non-uniform X-lattice.

The lattices in higher rank Lie groups have been studied extensively, with most definitive results obtained by G. A. Margulis. These results largely motivated the questions asked about the existence, structure and properties of tree lattices. The different behavior of rank-1 lattices from that of higher rank lattices suggests that we should prepare for new and unexpected answers for tree lattices in general. Below we will briefly survey and compare the properties of tree lattices, rank-1 lattices and higher rank lattices in Lie groups. For a more extensive survey consider [Lu2] and Chapter 0 in *Tree lattices*. We will denote by G a connected simple algebraic group defined over a local field k and of k-rank ≥ 2. It acts on the associated symmetric space X if k is archimedean and on the Bruhat-Tits building, also denoted by X, if k is not archimedean. A lattice Γ in G is a discrete subgroup with $\Gamma\backslash G$ of finite volume under the measure induced by Haar measure of G. A rank-1 simple Lie group will be denoted by H, and a general group of tree automorphisms will be denoted by $\mathcal{G}$. We will label by (G), (H), $(\mathcal{G})$ the information

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pertaining respectively to the case of lattices in higher rank Lie groups, rank-1 Lie groups and groups of tree automorphisms.

1. Existence of lattices

(G): A. Borel [Bo] showed that G has uniform and non-uniform lattices. G. Margulis showed that all higher rank lattices are arithmetic.

(H): If $charF = 0$, every lattice in H is uniform [Ta] and H has arithmetic [BH] and non-arithmetic [Lu1] lattices. If $charF = p > 0$ there exist non-uniform arithmetic lattices in H. There exist uniform arithmetic lattices if and only if H is of type (A) [BH]. A. Lubotzky showed [Lu1] that there exist uncountably many conjugacy classes of uniform and non-uniform H-lattices and, hence, non-arithmetic lattices of both types.

($\mathcal{G}$): The following theorem about existence of uniform tree lattices was proved in [BK].

Theorem 1 (Uniform Existence Theorem). *The following conditions are equivalent, in which case we call X a uniform tree:*

- *$\mathcal{G}$ is unimodular, and $\mathcal{G}\backslash X$ is finite;*
- *there exists a uniform X-lattice.*

Let $\mathcal{H}$ be a closed subgroup of $\mathcal{G} = AutX$. We will denote by $\mathcal{G}_{\mathcal{H}}$ the group of deck transformations, $\mathcal{G}_{\mathcal{H}} = \{g \in \mathcal{G} | gx \in \mathcal{H}x, \forall x \in X\}$. The following tree lattice existence result, applicable to the non-uniform lattices as well, is proved in full by H. Bass, L. Carbone and G. Rosenberg and can be found in an appendix of the book under review. Many cases of this theorem are proved in Chapter 7.

Theorem 2 (Lattice Existence Theorem). *There exists an X-lattice $\Gamma \subset \mathcal{G}_{\mathcal{H}}$ if and only if $\mathcal{H}$ is unimodular and $Vol(H\backslash\backslash X) < \infty$.*

Here $\mathcal{H}$ is a locally compact group with Haar measure μ, and $Vol(\mathcal{H}\backslash\backslash X) = \sum_{x \in \mathcal{H}\backslash X} \frac{1}{\mu(\mathcal{H}_x)}$.

Let X be a uniform tree. One can ask whether this tree admits a non-uniform lattice as well as a uniform one. The answer is negative in general. We call the tree X *rigid* if the group $\mathcal{G}$ is discrete. The simplest infinite example of a rigid tree is the linear (2-regular) tree $X = X_2$ with $\mathcal{G} = D_\infty$. A rigid tree need not admit any lattices, but if it does, they must be uniform (see Section 3.5). More generally, if X_0 is the minimal $\mathcal{G}$-invariant subtree of X, then the rigidity of X_0 implies that all X-lattices are uniform. H. Bass and A. Lubotzky had previously conjectured that this is the only obstruction to the existence of non-uniform lattices on a uniform tree. This conjecture was proved in full by L. Carbone [Ca1], [Ca2]:

Theorem 3. *Suppose that $\mathcal{H}$ is unimodular, $\mathcal{H}\backslash X$ is finite and $\mathcal{G}_{\mathcal{H}}|X_0$ is not discrete; then there exists a non-uniform X-lattice $\Gamma \subset \mathcal{G}_{\mathcal{H}}$.*

Several cases of this theorem were verified earlier by H. Bass and A. Lubotzky and are presented in Chapter 8 of the book, since L. Carbone makes use of them in her proof.

The notion of an arithmetic lattice does not exist in $\mathcal{G}$. However, if one restates Margulis' arithmeticity theorem as: A lattice Γ is arithmetic if and only if its commensurator $C_G(\Gamma)$ is not discrete; then one can define a lattice of $\mathcal{G}$ to be arithmetic if $C_{\mathcal{G}}(\Gamma)$ is not discrete. The commensurator $C_G(\Gamma)$ of Γ in G is a subgroup of G consisting of elements $g \in G$ such that $\Gamma \cap g\Gamma g^{-1}$ is of finite index in both Γ and $g\Gamma g^{-1}$. It turns out that every uniform lattice in $\mathcal{G}$ is "arithmetic" [Li]; in fact $C_{\mathcal{G}}(\Gamma)$ is dense in $\mathcal{G}$. In Chapter 10 of *Tree lattices* one finds examples of non-uniform lattices with discrete as well as non-discrete commensurators.

2. The structure of $\Gamma\backslash X$ and volumes

(G): Since every lattice of a higher rank Lie group G is arithmetic, reduction theory implies that a fundamental domain of Γ in G is covered by a union of finitely many Siegel sets [Ma]. Let $V(G)$ denote the set consisting of real numbers $\mu(\Gamma\backslash G)$, where μ is a measure induced by Haar measure on G, as Γ runs over all lattices in G. Then $V(G)$ is discrete, and there is a positive number ϵ such that $\mu(\Gamma\backslash G) > \epsilon$ for all lattices Γ.

(H): The quotient graph $\Gamma\backslash X$ is obtained from a finite graph by attaching finitely many infinite rays ("cusps") [Ra], [Lu1]. The set $V(H)$ is a closed discrete subset of $\mathbb{R}$.

($\mathcal{G}$): In contrast with the previous two cases we have

Theorem 4 (All Quotients Theorem)**.** *Given any connected locally finite graph A, there exist a locally finite tree X and an X-lattice Γ such that $\Gamma\backslash X \cong A$.*

This theorem is proved in Chapter 4 of the book. Call an integer $D > 0$ a d-number if for all primes p,

$$p|D \Rightarrow \begin{cases} p \leq d & \text{and} \\ p = d & \Rightarrow p^2 \not| D. \end{cases}$$

If Γ is a uniform X-lattice, then $Vol(\Gamma\backslash\backslash X)$ is a rational number whose denominator is a d-number. For non-uniform X-lattices a great variety of volumes can be found already on the regular trees. The following theorem is proved in Chapter 4:

Theorem 5 (Arbitrary Real Volumes Theorem)**.** *Let X be a regular tree of degree $d \geq 3$. Given $v > 0$, there is an X-lattice Γ with $Vol(\Gamma\backslash\backslash X) = v$ and $\Gamma\backslash X$ a ray.*

G. Rosenberg generalized the arbitrary volume theorem to all uniform trees that admit a non-uniform lattice.

3. Structure of Γ

(G): If rank $G \geq 2$, by [Ka], [Ma] G and Γ have property (T). Discrete groups with property (T) are finitely generated. Since Γ is arithmetic, the reduction theory [Bo2] implies that Γ is finitely presented. A group is said to be *residually finite* if the intersection of its finite index subgroups is the identity. Finitely generated linear groups are residually finite. Hence every lattice in G is residually finite.

(H): A non-uniform lattice in H is not finitely generated [Be], [Lu1]. Every lattice Γ in H is residually finite [Lu1], [Se].

($\mathcal{G}$): It was shown in [Ba] that a group Γ is isomorphic to a uniform tree lattice if and only if Γ is finitely generated and virtually free. Hence also Γ is residually finite. If Γ is a non-uniform X-lattice, then Γ is not finitely generated. This Γ may or may not be residually finite. Both kinds of examples are given in the book.

4. Centralizers, normalizers and commensurators

(G) and (H): We will describe the situation for both of these cases at the same time for convenience using only notation of (H). The centralizer $Z_H(\Gamma)$ is finite, the quotient $N_H(\Gamma)/\Gamma$ of the normalizer $N_H(\Gamma)$ by Γ is finite, and we have Margulis' Alternative: either the commensurator $C_H(\Gamma)$ is discrete and Γ is not arithmetic, or $C_H(\Gamma)$ is dense in H and Γ is arithmetic.

$(\mathcal{G})$: If Γ is a uniform tree lattice, then $Z_{\mathcal{G}}(\Gamma)$ is finite, unless Γ is virtually cyclic [BK], the group $N_{\mathcal{G}}(\Gamma)/\Gamma$ is finite [BK] and $C_{\mathcal{G}}(\Gamma)$ is dense in $\mathcal{G}$ [Li]. In the non-uniform case the situation is more complex. Let Γ be an infinite non-uniform tree lattice; then $Z_{\mathcal{G}}(\Gamma)$ is a closed subgroup of $\mathcal{G}$. If Γ fixes no end of the tree X, then $Z_{\mathcal{G}}(\Gamma)$ is shown to be a direct product of finite groups in Proposition 6.7. Γ acts minimally (without proper Γ-invariant subtrees) if and only if $\mathcal{G}$ acts minimally on X, in which case $Z_{\mathcal{G}}(\Gamma) = 1$ (see Sections 5.12, 6.5). The subgroup $N_{\mathcal{G}}(\Gamma)$ is closed in $\mathcal{G}$, and $N_{\mathcal{G}}(\Gamma)/\Gamma$ is a profinite group (see Proposition 6.8). The commensurator $C_{\mathcal{G}}(\Gamma)$ may or may not be dense in $\mathcal{G}$. Both kinds of examples are given in Chapter 10. If Γ acts minimally on X and $C_{\mathcal{G}}(\Gamma)$ is dense in $\mathcal{G}$, then Γ must be residually finite.

5. Further developments

The methods developed by H. Bass and A. Lubotzky in their study of tree lattices were further extended by M. Burger and S. Mozes [BM1], [BM2], who investigated the structure of (irreducible) lattices on a product of two (or more) trees. One can consider this theory as a generalization of the theory of lattices in semisimple Lie groups. In the case of lattices on product of trees, one meets rigidity phenomena [BMZ] not present for tree lattices. This is similar to the case of lattices in the semisimple group $PSL_2(\mathbb{Q}_p) \times PSL_2(\mathbb{R})$ which satisfy Margulis' Superrigidity theorem, whereas the lattices in the simple group $PSL_2(\mathbb{R})$ may not. The investigation of M. Burger and S. Mozes led to construction of remarkable examples of uniform lattices on a product of two trees which are finitely presented, torsion-free, simple groups. Furthermore, these lattices are free amalgams of finitely generated free groups. The question of existence of such groups was raised by P. Newmann. In addition, these lattices also present the first example of finitely presented simple groups of finite cohomological dimension.

6. Comments about the book

The book under review is a research monograph aimed at an extensive study of tree lattices. The questions considered here are motivated by the remarkable theory of lattices in simple Lie groups. The authors investigate the existence, structure and properties of tree lattices, drawing parallels and contrasts with the situation for lattices in Lie groups. The book, however, does not require knowledge of Lie theory and is essentially self-contained. The study of uniform tree lattices was initiated in [BK]. The present work focuses much more on the non-uniform lattices, for which the phenomena are considerably more complex. In this way the book can be considered a sequel to [BK].

The methods the authors use are based on the notion of graphs of groups first developed by J.-P. Serre [Se] and edge indexed graphs. These techniques were further elaborated upon in [Ba] and [BK] and are reviewed in the text (Chapter 2).

The book contains a great number of examples that are extremely helpful in motivating and illustrating the results proved. There are three appendices. The first appendix, by H. Bass, L. Carbone and G. Rosenberg, contains the complete proof of existence of tree lattices whenever the group $\mathcal{G} = Aut(X)$ is unimodular and $\mathcal{G}\backslash X$ has finite volume. The second appendix, by H. Bass and J. Tits, presents a criterion for the full automorphism group of a tree to be discrete. Such a group cannot contain a non-uniform tree lattice. The third appendix, by H. Bass and A. Lubotzky, describes a group theoretic construction of P. Newmann which was used to produce some interesting examples of self-normalizing non-uniform lattices. In fact, the book comments on or presents the results of the research of so many mathematicians in the field that apparently it was difficult to find an uninvolved person to review the book. On the other hand, it makes this book the most complete and up to date reference in the theory of tree lattices.

In summary, I found this book extremely well written and enjoyable, with an abundance of very helpful examples clarifying the theory. I recommend this book highly to the interested reader.

References

[Ba] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89**(1993), 3-47. MR **94j:**20028

[BK] H. Bass and R. Kulkarni, *Uniform tree lattices*, J. Amer. Math. Soc. **3**(1990), 843-902. MR **91k:**20034

[BL] H. Bass and A. Lubotzky, *Tree lattices*, with appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits. Progress in Mathematics, **176**, Birkhauser Boston Inc., Boston, 2001. MR **2001k:**20056

[Be] H. Behr, *Finite presentability of arithmetic groups over global function fields*, Proc. Edinburgh Math. Soc. **30**(1987), 23-39. MR **88f:**11032

[Bo] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2**(1963), 111-122. MR **26:**3823

[Bo2] A. Borel, *Introduction aux groupes arithmetiques*, Hermann, Paris, 1969. MR **39:**5577

[BH] A. Borel and G. Harder, *Existence of discrete cocompact subgroups of reductive groups over local fields*, J. Reine Angew. Math. **298**(1978), 53-64. MR **80b:**22022

[BM1] M. Burger and S. Mozes, *Lattices in product of trees*, Inst. Hautes Etudes Sci. Publ. Math., no.92(2000), 151-194(2001). MR **2002i:**20042

[BM2] M. Burger and S. Mozes, *Groups acting on trees: from local to global structure*, Inst. Hautes Etudes Sci. Publ. Math. no.92(2000), 113-150(2001). MR **2002i:**20041

[BMZ] M. Burger, S. Mozes and R. Zimmer, *Irreducible lattices in the automorphism group of a product of trees, superrigidity and arithmeticity*, in preparation.

[Ca1] L. Carbone, *Non-uniform lattices on uniform trees*, Mem. Amer. Math. Soc. **152**(2001), no. 724. MR **2002k:**20045

[Ca2] L. Carbone, *Non-minimal actions and the existence of non-uniform tree lattices*, in preparation.

[Ka] D. A. Kazhdan, *On connection between the dual space of a group and the structure of its closed subgroups*, Funct. Anal. Appl. **1**(1967), 63-65.

[Li] Y. S. Liu, *Density of the commensurability groups of uniform tree lattices*, J. Algebra **165**(1994), 346-359. MR **95c:**20036

[Lu1] A. Lubotzky, *Lattices in rank one Lie groups over local fields*, Geom. Funct. Anal. **1**(1991), 405-431. MR **92k:**22019

[Lu2] A. Lubotzky, *Tree-lattices and lattices in Lie groups*, Combinatorial and Geometric Group Theory (Edinburgh, 1993) 217-232, London Math. Soc. Lecture Notes Ser., **204**, Cambridge Univ. Press, Cambridge, 1995.

[Ma] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, Berlin, 1991. MR **92h:**22021

[Ra] M. S. Raghunathan, *Discrete subgroups of algebraic groups over local fields of positive characteristics*, Proc. Indian Acad. Sci. Math. Sci. **99**(1989), 127-146. MR **91a:**22010

[Se] J. P. Serre, *Trees*, Springer-Verlag, New York, 1980. MR **82c:**20083

[Ta] T. Tamagawa, *On discrete subgroups of p-adic algebraic groups*, in Arithmetical Algebraic Geometry, O.F.G. Schilling (ed.), Harper and Row, New York, 1965, 11-17. MR **33:**4060

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