BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, Pages 259–262
S 0273-0979(03)00980-7
Article electronically published on February 12, 2003

Theory of difference equations — numerical methods and applications, 2nd ed., by V. Lakshmikantham and Donato Trigiante, Marcel Dekker, Inc., New York, 2002, x+300 pp., \$150.00, ISBN 0-8247-0803-2

The idea of computing by recursion is as old as counting itself. It occurred in primitive form in the efforts of the Babylonians as early as 2000 B.C. to extract roots and in more explicit form around 450 B.C. in the Pythagoreans' study of figurative numbers, since in modern notation the triangular numbers satisfy the difference equation $t_n = t_{n-1} + n$, the square numbers the equation $s_n = s_{n-1} + n^2$, and so forth. The Pythagoreans also used a system of difference equations $x_n = x_{n-1} + 2y_{n-1}$, $y_n = x_{n-1} + y_{n-1}$ to generate large solutions of Pell's equation, $x^2 - 2y^2 = 1$, and thereby approximations of $\sqrt{2}$. In his attempts to compute the circumference of a circle, Archimedes (about 250 B.C.) employed equations of the form $P_{2n} = 2p_nP_n/(p_n + P_n)$, $p_{2n} = \sqrt{p_nP_{2n}}$ to compute the perimeters P_n and p_n of the circumscribed polygon of n sides and the inscribed polygon of n sides, respectively. Other familiar ancient discoveries about recurrence included the Euclidean algorithm and Zeno's paradox. Euclid also studied geometric series, although the general form of the sum was not obtained until around 1593 by Vieta.

About 1202, Fibonacci formulated his famous rabbit problem that led to the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \cdots$. However, it appears that the corresponding difference equation $F_n = F_{n-2} + F_{n-1}$ was first written down by Albert Girard around 1634 and solved by de Moivre in 1730. Bombelli studied the equation $y_n = 2 + 1/y_{n-1}$ in 1572, which is similar to the equation $z_n = 1 + 1/z_{n-1}$ satisfied by ratios of Fibonacci numbers, in order to approximate $\sqrt{2}$. Fibonacci also gave a rough definition for the concept of continued fractions that is intimately associated with difference equations. A more precise definition was formulated by Cataldi around 1613. (See Brezinski [2] for a lively discussion of the history of continued fractions.) The earliest known example of a difference equation in two indices, namely the equation $b_{n+1,r} = b_{n,r} + b_{n,r-1}$ for the binomial coefficients, can be traced back to Chia Hsien (1050?) and Omar Khayyam (1100?). The method of recursion was significantly advanced with the invention of mathematical induction by Francesco Maurolico in the sixteenth century and with its development by Fermat and Pascal in the seventeenth century.

Sir Thomas Harriet (1560-1621) invented the calculus of finite differences, and Henry Briggs (1556-1630) applied it to the calculation of logarithms. It was rediscovered by Leibniz around 1672. Newton, Euler, LaGrange, Gauss, and many others used this calculus to study interpolation theory. The theory of finite differences was developed largely by Stirling in the early eighteenth century. Goldstine [8] gives a detailed historical description of the early work in this area. Meanwhile, an important class of nonlinear difference equations, which we now call Newton's method (known in primitive form by Vieta), was used by Newton around 1669 to study solutions of $y^3 - 2y - 5 = 0$ and later in computations for Kepler's equation. In 1690, Raphson worked out a more systematic treatment of the method. Another important family of nonlinear difference equations consists of pairs of equations

2000 *Mathematics Subject Classification.* Primary 39A10.

involving arithmetic and geometric means. For example, LaGrange obtained the equations $x_n = (x_{n-1} + y_{n-1})/2$, $y_n = \sqrt{x_{n-1}y_{n-1}}$ as an algorithm for the reduction and evaluation of elliptic integrals. Gauss and Borhardt discovered related algorithms, and Gauss' investigations led him to the discovery of elliptic functions (see Carlson [3]).

The basic theory of linear difference equations was developed in the eighteenth century by de Moivre, Euler, Lagrange, Laplace, and others. Generating functions, first used by de Moivre to solve the Fibonacci equation, were exploited by Laplace as part of his work in probability theory. He also introduced integral representations of solutions and their asymptotic behavior (Laplace's method). Actually, asymptotic analysis entered the subject somewhat earlier when Maclaurin (1742), and independently Euler (1755), stymied by the dearth of closed formulas for sums, discovered recipes for approximate summation. The foundation for a thorough study of the asymptotic properties of solutions of linear difference equations was laid in the 1880's by Poincaré, who formalized the concept of asymptotic series and also showed that under favorable conditions the ratio of consecutive values of a solution must approach a characteristic root. In 1909, Perron gave a significant extension of this result, and the asymptotic theory was brought to a certain level of completeness by Birkhoff and his students in 1930.

The idea of using difference equations to approximate solutions of differential equations originated in 1769 with Euler's polygonal method, for which the proof of convergence was given by Cauchy around 1840. The subject seems to have languished until almost the end of the nineteenth century, when Lipschitz, Runge, and Kutta developed improved procedures. The urgent need for numerical approximations during World War I greatly stimulated research in this area, and the number of publications later exploded with the development of the digital computer. Dahlquist [4] initiated the modern theory of the convergence of multistep methods (see Dahlquist [5] for a historical account).

The efficient application of linear difference equations to the computation of special functions orginated in 1952 with Miller's algorithm for Bessel functions. Such computations must be done with care because of the possibility of explosive roundoff error, as illustrated by the cautionary example of Gautschi [6]. Wimp [10] discusses the development of this method and related algorithms due to Olver, Clenshaw, Gautschi, Wimp, and others, as well as some examples of computation with nonlinear difference equations. Further development of the theory of linear difference equations has brought the subject to a state comparable to that of linear differential equations (see Hartman [9], and Ahlbrandt and Peterson [1]).

During the 1950's, several ecologists used simple nonlinear difference equations, including the logistic equation, to study the change in populations from one year (or season) to the next with the emphasis on the stability of the iteration. However, in the early 1970's May investigated the variety of complex behavior exhibited by the logistic equation and pondered the possible relationship of this behavior to observed fluctuations in real populations. Additional discoveries about the logistic and related equations were soon made by York, Sarkovskii, Feigenbaum, and others (see Gleick [7]), and the remarkably intricate properties of these equations led to their becoming a focus in the developing area of chaotic dynamical systems. The excitement of these discoveries attracted the attention of researchers who attempted to apply the results to fields from economics to medicine.

Even a brief survey of the history of computing with recurrences, such as that sketched above, indicates the breadth of the area. The book under review addresses a reasonable portion of the subject by concentrating on the basic properties of linear difference equations, stability theory for linear and nonlinear equations, and several applications to numerical analysis and other fields. It is intended to be a textbook for graduate students, and most chapters contain a nice range of suitable exercises.

The introductory material on the discrete calculus and related topics in the text is interesting because it features the Pascal matrix and its relationship to Bernoulli polynomials, Stirling numbers, Bernstein polynomials, and the Vandermonde matrix. This approach yields elegant derivations of several combinatorial identities. The treatment of linear equations with constant coefficients is standard and includes basic facts about generating functions and a brief mention of z transforms. The basic theory of systems of linear equations is also presented, along with a matrix proof of Poincaré's Theorem. Perron's Theorem is stated without proof, and there is little indication of how these theorems might be applied.

Chapter 4 is perhaps the most comprehensive discussion of stability theory to be found in books on difference equations. At least thirteen types of stability are defined, including the concepts of total and practical stability that are employed later in the text in results about numerical analysis. Relationships between different types of stability are examined for linear and nonlinear equations, and the important topic of the domain of asymptotic stability is included. There is also a chapter that explores the relationship between difference equations and banded matrices. It shows how various important subjects, such as the properties of orthogonal polynomials, the use of the Euclidean algorithm to find roots of a generic polynomial, and cyclic reduction, can be studied in this context. Several interesting nonlinear difference equations are generated by these calculations.

Stability theory is used to establish convergence of various iterative methods for approximating roots of equations. Although most of these results are local in nature, semilocal results can also be obtained by exploiting the possible contractivity of the governing function. There is a very brief treatment of Miller's and related algorithms for computing minimal solutions of linear difference equations. This material might have been included in expanded form in the earlier chapter on banded matrices. Among the applications discussed in the book, the chapter on numerical methods for differential equations is the most interesting. It is shown how stability techniques for difference equations can be used to study the convergence of multistep methods for ordinary differential equations. Also, convergence of the method of lines is established for two problems for partial differential equations. Real world phenomena that are modeled via difference equations include population dynamics (Leslie model, population waves, complex behavior of the logistic equation), distillation of an ideal mixture of two liquids using a column of plates, cobweb model in economics, and queuing theory. Most of the equations treated are linear and, consequently, do not involve much variety in behavior. The last chapter in the book presents a number of historically important difference equations.

The book is marred somewhat by a looseness of presentation and would benefit from a good proof-reading. For example, on page 105 the authors claim that $\lim_{n\to\infty} \Pi_{i=n_0}^{n-1} |\cos i| \neq 0$, Theorem 4.8.6 needs $\omega \geq 0$, the statements of Definition 4.11.2 and Theorem 4.8.5 are confusing, equation (6.16) is incorrect, and on page 241 Euclid should be Euler. The discussion in Chapter 7 would be clearer if presented in theorem form. There are many awkward English constructions, especially in Chapter 5. Nevertheless, this book offers a valuable and distinctive contribution to the literature on difference equations and numerical analysis. It could serve as a useful textbook for beginning graduate students of applied mathematics.

References

[1] C. Ahlbrandt and A. Peterson, Discrete Hamiltonian Systems, Kluwer, Dordrecht, 1996. MR **98m:**39043

[2] C. Brezinski, History of Continued Fractions and Padé Approximants, Springer-Verlag, New York, 1991. MR **92c:**01002

[3] B. Carlson, Algorithms involving arithmetic and geometric means, AMS Monthly 78(1971), 496-505. MR **44:**479

[4] G. Dahlquist, Convergence and stability in the numerical integration of ordinary differential equations, Math. Scandinavica 4(1956), 33-50. MR **18:**338d

[5] G. Dahlquist, 33 years of numerical instability, part 1, BIT 25(1985), 188-204. MR **86h:**65006

[6] W. Gautschi, Zur Numerik rekurrenter Relationen, Computing 9(1972), 107-126. MR **47:**1270

[7] J. Gleick, Chaos: Making a New Science, Penguin Books, New York, 1987. MR **91d:**58152

[8] H. Goldstine, A History of Numerical Analysis from the 16th through the 19th Century, Springer-Verlag, New York, 1977. MR **58:**4774

[9] P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete montonicity, Trans. AMS 246(1978), 1-30. MR **80a:**39004

[10] J. Wimp, Computation with Recurrence Relations, Pitman, Boston, 1984. MR **85f:**65001

Walter Kelley
University of Oklahoma
E-mail address: wkelley@ou.edu