BOOK REVIEWS

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Lie groups: An introduction through linear groups, by Wulf Rossmann, Oxford Grad. Texts Math., vol. 5, Oxford Univ. Press, Oxford, 2002, x+265 pp., \$60.00, ISBN 0-19-859683-9

The best theories are the ones that have settled, either by virtue of their actual genesis or more commonly through their subsequent evolution, at the right level of generality. They must be sufficiently general to encompass problems of broad interest and applicability, but not so super-general as to allow for an expanse of phenomena not amenable to any sort of reasonable taxonomy. There is of course a litany of subjects of questionable merit that fail to satisfy one of these criteria (and sometimes both), but in these terms it is difficult to imagine one that meets them more spectacularly than the representation theory of finite-dimensional simple Lie algebras. The definition is entirely elementary: a Lie algebra is a vector space $\mathfrak{g}$ together with an alternating bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity (a kind of associativity for $[\cdot,\cdot]$). The modifier "simple" means that $\mathfrak{g}$ is atomic in the sense that it has no proper nontrivial ideals; here an ideal is a subspace such that $[X, Y]$ also belongs to the ideal whenever X is in the ideal and Y is an arbitrary element of $\mathfrak{g}$. (Conventionally a simple Lie algebra is also not allowed to have dimension one.)

A hint at why this definition is so interesting is the classification of simple complex Lie algebras (achieved in 1894 by E. Cartan). Since there are no ideals to help, one works with the next best thing: a maximal abelian subspace $\mathfrak{h}$ (i.e. a subspace on which the restriction of $[\cdot,\cdot]$ is identically zero). It turns out that each $[H,\cdot] \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ for $H \in \mathfrak{h}$ is a semisimple endomorphism of $\mathfrak{g}$, and since $\mathfrak{h}$ is abelian, it is possible to simultaneously diagonalize the action of each $[H,\cdot]$. The resulting nonzero generalized eigenvalues $\alpha \in \mathfrak{h}^*$ are called roots. They satisfy a strong rationality property, and as a consequence their $\mathbb{Z}$-span is a lattice of rank equal, in fact, to the full dimension of $\mathfrak{h}^*$. This so-called root lattice essentially characterizes $\mathfrak{g}$ and (among other things) comes equipped with a Euclidean structure. The notion of a root lattice can be axiomatized, and subsequently all such can be classified and each can be shown to correspond to a complex simple Lie algebra. The classification yields four infinite families of algebras, each more or less indexed by the natural numbers (for instance, one family is the set of traceless endomorphisms of $\mathbb{C}^n$ with $[A, B] = AB - BA$), and five "exceptional" algebras,

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G_2, F_4, E_6, E_7, and E_8, where the subscript indicates the dimension of the root lattice.

Regardless of whether or not the details of this classification are familiar to the reader, the point is that the answer foreshadows the flavor of the entire subject. It is discrete in essence, and consequently this combinatorial backbone permeates many aspects of the representation theory of simple Lie algebras, even where one might not expect much combinatorics at all. As for the specifics of the classification, the existence of such a short list of exceptional algebras is completely unexpected: one would either expect no exceptional algebras at all or a very long list of them. They must surely be considered a gift of divinity gratis. It should come as no surprise, therefore, that the root lattice for E_8 is one of the world's most interesting lattices. For instance, it is the best lattice packing of spheres in eight dimensions, probably the best of *any* packing in eight dimensions (though this is still open); and in terms of a canonical normalization of distance, the number of lattice points whose length is less than $2n$ (for any integer n) is equal to 240 times the sum of the cubes of the exact divisors of n.

But the classification is of course just the tip of the iceberg. Cartan went on to study the irreducible finite-dimensional representations of a complex simple Lie algebra $\mathfrak{g}$. Such a representation is a linear map $\phi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ respecting the bracket operator in the sense that $\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$. It is irreducible if there is no proper nontrivial subspace $V' \subset V$ such that $\phi(\mathfrak{g})V' \subset V'$. There is a natural notion of a choice of a positive system of roots Δ^+ such that each root α is either in Δ^+ or else $-\alpha \in \Delta^+$. For a fixed choice of Δ^+, it transpires that each irreducible finite-dimensional representation contains a vector v, unique up to scalar, such that $\phi(X)v = 0$ for all $X \in \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$; here $\mathfrak{g}^\alpha$ is the generalized α eigenspace for the action of $[H,\cdot] \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$. Moreover, there exists $\lambda \in \mathfrak{h}^*$ such that $\phi(H)v = \lambda(H)v$ for all $H \in \mathfrak{h}$. (The terminology is that v is a highest weight vector of weight λ.) The weight λ characterizes (the equivalence class of) the irreducible representation, and it is easy to say which λ arise as highest weights of finite-dimensional irreducible representations. This is the celebrated Theorem of the Highest Weight, and it has ubiquitous applications.

The finite-dimensional highest weight theory is really all that is germane to the current review, but having come this far it is impossible to resist at least giving a taste of the beautiful theory of infinite-dimensional irreducible representations of $\mathfrak{g}$ which possess a highest weight vector in the sense of the preceding paragraph. Suitably defined (as in the fundamental work of Bernstein-Gelfand-Gelfand), these representations form a category $\mathcal{HW}$, and the classification of irreducible objects follows in much the same way as the finite-dimensional theory. But here a new wrinkle enters: there are nontrivial extensions between irreducible objects in $\mathcal{HW}$, something that does not happen for finite-dimensional irreducibles. In the late 1970's, Kazhdan and Lusztig changed the trajectory of inquiry by introducing an auxiliary algebra and suggesting (in a slightly different form) that the extension problem possibly had purely geometric content. Subsequently it was established that $\mathcal{HW}$ was equivalent, very roughly speaking, to a geometric category of equivariant sheaves. The irreducible objects in $\mathcal{HW}$ corresponded to sheaves supported on singular algebraic varieties, and the computation of Ext groups in $\mathcal{HW}$ amounted to computing the local intersection cohomology of these singular spaces.

The development of the theory of highest weight modules proved to be a paradigm for other categories of representations, and many questions regarded once as purely representation theoretic have subsequently been interpreted as problems in the study of the homology of singular algebraic varieties. Perhaps the most mysterious of all is the geometric theory of the Langlands conjectures, where even questions of arithmetic harmonic analysis have been conjecturally translated into purely geometric ones. In some narrowly interpreted cases, representation theory has been subsumed as a subfield of algebraic geometry—a subfield, incidentally, which is arguably of the right level of generality in the sense discussed at the outset.

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For Lie, the progenitor of the subject, the fundamental objects of study were "finite continuous groups". Roughly speaking, Lie's finite continuous groups are what we nowadays call Lie groups, that is, a smooth manifold G together with a group structure whose multiplication and inversion maps are smooth.

The above discussion of simple Lie algebras — with no groups at all in sight — was only meant to touch on some ideas in the subject. (To mention only two serious omissions: there is no mention of the vital contributions of Weyl to the highest weight theory, and there is no mention of the Borel-Weil theorem describing geometric realizations of finite-dimensional representations, even though this theorem was decisive motivation for the kind of geometric categorical equivalence mentioned in the context of $\mathcal{HW}$ above.) But the conspicuous absence of groups in the discussion was not entirely without calculation. The key point is that understanding the definition of simple Lie algebras, their classification, the essence of the highest weight theory, and even the original formulation of the Kazhdan-Lusztig conjecture requires nothing beyond a command of abstract linear algebra. But even the definition of a Lie group requires substantially more, including at least an understanding of manifolds, point-set topology, and abstract group theory.

To understand the further pedagogical implications of this observations, one needs to understand the so-called Lie correspondence, the dictionary between Lie groups and Lie algebras. To each Lie group G, one associates a real Lie algebra of vector fields that are invariant under left translation. This construction is local, so it cannot distinguish the connected component of the identity of G from G itself, nor can it distinguish G from a topological covering of G. But it is relatively easy to make precise the sense in which this construction is one-to-one. (That it is onto — that is, that every real Lie algebra comes from a Lie group — is a difficult result often known as Lie's Third Theorem.)

One may ask to what extent the passage from a Lie group G to Lie algebra $\mathfrak{g}$ behaves nicely with respect to subgroups and subalgebras. This is not so easy. Intuitively the kind of statement that one wants is that the Lie subalgebras of $\mathfrak{g}$ correspond exactly to the Lie subgroups of G. It is clear what one means by a Lie subalgebra (a vector subspace closed under bracket), but exactly what one means by a Lie subgroup is more subtle. A basic example is the two-torus with Lie algebra $\mathbb{R}^2$, where one wants a line of irrational slope in $\mathbb{R}^2$ to correspond to an infinite winding on the torus. This example shows that one must consider subgroups (like the infinite winding) whose topology does not coincide with the relative topology on the whole of G. On the other hand, the closed subgroup theorem asserts that a *closed* subgroup of a Lie group with the relative topology is again a Lie group, and it follows easily that the Lie algebra of the subgroup is a subalgebra of the Lie algebra of the ambient group. The issue then is to find some class of subgroups of G that contains the closed ones and which corresponds exactly to subalgebras of $\mathfrak{g}$. Chevalley found the right class of subgroups, and this is one of the main contributions of his classic 1946 text [Ch]. To establish the correspondence of subalgebras and subgroups, he relied on the Frobenius theory to solve systems of partial differential equations defining submanifolds, or, in a more geometric language, to pass from involutive subbundles of the tangent bundle of a given manifold to integral submanifolds.

Finally, one may ask to what extent morphisms of Lie groups correspond to morphisms of their associated Lie algebras. By differentiation a smooth map of Lie groups gives a map of corresponding Lie algebras. While each map of Lie algebras need not lift to the groups from which they come, the map does at least lift to the simply connected coverings of the groups in question. These statements follow relatively painlessly (using an argument in [Ch]) from the correspondence of subgroups and subalgebras given in the previous paragraph.

It is worth pausing to emphasize the remarkable nature of the Lie correspondence. It makes precise the sense in which the structure of a Lie group is so rigid that the group is essentially controlled by its Lie algebra, an ostensibly far less structured object. In turn, we sketched above how the most basic and atomic Lie algebras (the simple ones) are essentially controlled by a combinatorial structure. Thus a simple Lie group — an outwardly complicated confluence of geometric, analytic, and algebraic structures — remarkably reduces in many ways to a kind of combinatorial skeleton. This astounding reality at once captures the flavor and power of the subject.

To return to more immediate matters, the above discussion clearly indicates that understanding the dictionary between Lie groups and Lie algebras demands significant mathematical sophistication beyond even that required to understand and appreciate the definition of a Lie group. The consequent pedagogical problem is that a linear treatment of the Lie correspondence, together with the time it takes to fill in any missing prerequisites, can easily burn almost an entire semester's course. Students leave with the dictionary in hand along with a few examples of Lie groups but little else except possibly a painful memory of a tortuously dry experience and possibly a too-hurried overview of some aspects of the highest weight theory. Increasingly it seems that graduate students (and strong undergraduates) are adapting to this reality by learning the more elementary Lie algebra theory up to the highest weight theory (say from the standard and still excellent [Hu]), accepting the dictionary as axiomatic (and without proof), and then applying the theory of Lie groups to the particular setting that they need in their own thesis work, be it in symplectic geometry or mathematical physics or whatever their specialty may be. In principle there is nothing wrong with this approach. After all, most learning in graduate school follows such a nonlinear path where some concepts are taken on faith in the kind of synthesis that underlies all mathematical research. But while good students of mathematics always return to fill in their missing proofs, less-than-good students do not. The danger is the creation of a generation of practitioners of Lie theory who are ignorant of the foundations of the theorems they employ.

Of course if there were an expository breakthrough making the dictionary at once more accessible and coherent, the danger might be avoided altogether. Certainly people have tried. The review [K] counts over twenty-five attempts, none of which deserves the title of breakthrough in this sense. One must ask: Is it really reasonable to expect a significant improvement (beyond incremental polishing and updating) of the gold standard of Chevalley's treatment of the Lie correspondence? Rossmann's book provides an affirmative answer.

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In an effort to simplify the exposition of the correspondence of Lie groups and Lie algebras, a natural idea is to focus not on general Lie groups but instead on linear groups, that is, those Lie groups whose elements consist of invertible matrices. This is a nontrivial restriction, yet a restriction that makes good sense: it is linear groups that arise most frequently in applications and capture the subtleties of the theory, and the general nonlinear theory can best be absorbed after a mastery of linear examples. This is the approach that Rossmann's book takes.

But even after agreeing on narrowing the focus to groups of matrices, further choices must be made for the subsequent exposition of the Lie correspondence. The essence of the further choices is topological. Chevalley's argument relating subgroups to subalgebras makes essential use of the manifold structure of the objects under consideration (inasmuch as he invokes the Frobenius theory). Thus most treatments of linear groups from the very beginning restrict attention further to those that have a manifold structure. (Anticipating the closed subgroup theorem — which can be proved from rather elementary principles, as in [Ho] for example — some treatments are confined to closed subgroups of matrices, but as mentioned above, this is too restrictive to treat the subgroup-subalgebra correspondence.) As Rossmann aptly points out in the preface, the imposition of additional topological hypotheses appears logically superfluous, since ultimately it is often deduced after the fact that any connected abstract subgroup of invertible matrices is indeed a manifold in a natural way. But any argument involving the Frobenius theorem is such that this consequence cannot be taken as the starting point.

Rossmann adopts a new approach and instead works with arbitrary sets of invertible matrices closed under inversion and multiplication but with absolutely no topological assumptions. In the first chapter he develops the main tool of his treatment, the exponential map of matrices, which is defined as usual as a power series. In a neighborhood of the identity matrix in the space of all invertible matrices of a fixed size, the exponential map admits an inverse. In this neighborhood it is possible to relate the product $\exp(X)\exp(Y)$ to iterated bracket operations of the form $XY - YX$. This is the Campbell-Baker-Hausdorff formula, which appears at the end of Chapter 1 and which plays a decisive role in the subsequent exposition.

The second chapter defines the Lie algebra $\mathfrak{g}$ of a linear group G, as usual, as the set of matrices that appear as differentials (at the identity) of curves in G. It is easy to see that $\mathfrak{g}$ is closed under the bracket operation $XY - YX$, so really defines a Lie algebra. But then Rossmann, again with no additional topological assumptions on G, establishes a surprising result (or at least surprising to me): the exponential map always carries the Lie algebra $\mathfrak{g}$ back into G. Given this fact, he can import a topology to G by declaring a basis of open sets to be the translates in G of the exponential image of ϵ-balls in $\mathfrak{g}$ (defined by the requirement that the sum of the squares of their matrix entries are less than ϵ). This is a nonstandard topology and need not agree with the relative topology on G inherited from the full space of invertible matrices. The fact that $\exp(\mathfrak{g}) \subset G$ also allows one to import coordinates on G by exponentiating a basis of $\mathfrak{g}$. This gives the connected component of the identity in G the structure of a manifold (though Rossmann does not yet use this terminology).

With the treatment of coordinates and the Campbell-Baker-Hausdorff formula in hand, Rossmann is able to prove the correspondence between subalgebras of $\mathfrak{g}$ and subgroups of G that are connected in their intrinsic (not relative) topology. The correspondence is achieved in Section 2.5 by a lovely conceptual argument. From there, Rossmann adapts an argument originating in [Ch] and gives a clean treatment of the correspondence of morphisms. Even here his exposition is more elementary than one might have thought possible and avoids any recourse to the definition of homotopy groups. Finally, he proves the closed subgroup theorem for linear groups in the guise that the relative and nonstandard topologies of a closed subgroup of a linear group coincide.

Thus in the span of just two chapters, weighing in at a not-very-dense ninety pages, Rossmann provides nearly the entire dictionary between linear Lie groups and their Lie algebras. (There is no treatment of Lie's Third Theorem, but this is too difficult for any elementary book on the subject.) Nothing beyond linear algebra, rudimentary abstract group theory, and multivariable calculus (up to the inverse function theorem) is required. In particular, there is no mention of manifolds, no use of the Frobenius theorem, and no mention of homotopy groups. This is a significant accomplishment.

The remaining four chapters of the book expand the scope of the Lie correspondence to general Lie groups, discuss the classical real Lie groups in detail, and treat the highest weight theory (as sketched above) for finite-dimensional representations of classical groups. The treatment of these topics is more in line with other existing sources, though it is notable that Rossmann works not to duplicate these sources but to complement them (and always with an eye for keeping prerequisites minimal). Typical of this approach is Section 3.4. Here he at last defines the fundamental group and, following Weyl, gives a beautiful and visually intuitive computation of π_1 of the compact classical groups. Rossmann concludes the section with Cartan's version of the computation, which is based on the theory of the root lattice (and which generalizes to exceptional groups). Rossmann avoids the standard inductive computation using the long exact sequence of homotopy groups, and this is appropriate for at least three good reasons in the current context: the inductive argument is less conceptual, it requires more machinery, and (unlike the approaches of Weyl and Cartan) it appears in a great many other sources.

The book would have benefited enormously from more careful editing. There are an alarming number of typographical errors, including extraneous symbols inserted apparently at random, characters appearing in the wrong font type, and references to nonexistent entries in the bibliography. These are mostly innocuous, sometimes annoying, and at least on one occasion actually disruptive. The disruption comes in the form of a whole series of confusing misprints in the proof of Theorem 1 beginning on page 67. (The placement of these misprints is especially unfortunate, as this theorem is really at the heart of Rossmann's unique approach.) The review [K] carefully details these mistakes and corrects them. Rossmann also maintains a webpage, easily accessible from his homepage, where he has begun documenting corrections to the text.

Anyone who has been privileged to work with Rossmann's research ideas knows that he is beholden to no one's traditions but his own. In capable hands, that kind of free thinking invariably leads to considerable and unanticipated advances. This time Rossmann has turned his attention to the exposition of elementary Lie theory and has, indeed, advanced the subject considerably.

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Peter E. Trapa
University of Utah
E-mail address: `ptrapa@math.utah.edu`