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The symmetry perspective: From equilibrium to chaos in phase space and physical space, by Martin Golubitsky and Ian Stewart, Birkhäuser Verlag, Basel, Boston and Berlin, 2003, xvii + 325 pp., 44.94 euros, CHF 68 (softcover), ISBN 3-7643-2171-7; 75 euros, CHF 113 (hardcover), ISBN 3-7643-6609-5

Attempts to apply mathematics to questions in science or technology often suffer from the problem that an analytical solution of such practical questions is usually either trivial (at least for anyone sufficiently well trained) or impossible (even for the most expert practitioners of mathematics of any given age). In applications of nonlinear dynamical systems this manifests itself as the unfortunate reality that typical dynamical systems, whether expressed as maps or differential equations, usually cannot be solved or even usefully approximated in any analytical form whatsoever. In the absence of explicit solutions, one needs to apply a range of qualitative and asymptotic techniques. More recently, the growth in speed and availability of computers for approximation and simulation has allowed researchers to gain an understanding of the typical behaviour of a wide range of dynamical systems that would have previously been inaccessible to study by either explicit or qualitative methods.

Since mathematical models from applications often include symmetries, a natural question is how symmetries of the model manifest themselves in the asymptotic dynamics and bifurcations thereof. This naturally leads to the study of equivariant dynamical systems: flows and maps that commute with an action of a symmetry group. Several areas of applied mathematics, theoretical physics and chemistry have successfully used this approach, going back to Wigner's work on atomic spectra. In particular, stabilities of equilibria are generically determined by linearizations; these linearizations must commute with the symmetry group, and one can readily split the spectral properties of the linearization into parts according to a decomposition of the action of the group into copies of irreducible representations. However, the usefulness of the group action goes beyond that; it constrains the normal forms that describe the bifurcation of solutions at a center manifold. Some terms are forced to be zero, while others are generically nonzero. It also forces the existence of invariant sets purely by virtue of their symmetries. The main mathematical tools used in this approach consist of Lie group representation theory, equivariant singularity theory for smooth germs of vector fields and adaptation of other methods from dissipative dynamics and bifurcation theory [4], [1] to an equivariant setting.

The book presently under review can be seen as a continuation of the research programme of the authors that was gathered together previously into two volumes [2], [3]. However, the present book is not really intended as an updated exposition or extension of the theory in those volumes. Instead it brings in developments of the theory as necessary and as inspired by a number of applications that are discussed throughout the book.

The applications could be classified loosely into problems of *pattern formation* and problems of *nonlinear oscillation*.[1] For example, pattern formation problems

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[1]One could also divide these problems into dissipative and conservative (Hamiltonian) dynamics depending on whether or not there are smooth quantities or other features such as symplectic

arise in Rayleigh-Bénard convection, fluid flow or biological systems [5] and typically consist of problems expressed as parabolic partial differential evolution equations where one would like to know which spatio-temporal patterns are possible, which are stable and how they are connected on varying a system parameter. Nonlinear oscillation problems arise in many physical and biological systems and are typically expressed as coupled ordinary differential equations. In these problems one would like to understand the process of formation of spatio-temporal patterns of oscillation, manifesting themselves as collective behaviour of coupled dynamical units (a good introduction for the general reader is [7]). For both pattern formation and nonlinear oscillation problems, one is interested in asymptotic properties of solutions to initial value problems and in particular properties of invariant sets; these may or may not possess symmetries of the system. One particular question addressed in Chapter 9 is precisely which symmetries are possible for attractors of a system with a given symmetry group. Surprisingly, this can give information about the dynamics; Theorem 3.4 characterizes the fact that periodic orbits can only have symmetries that are cyclic extensions of isotropy subgroups of the action, up to possible topological obstructions in low dimensions. More general attractors (Lyapunov stable sets that are ω-limits of some point) are characterized as having a larger range of possible symmetries by Theorem 9.3; they are restricted only by the presence of codimension one invariant subsets forced by the symmetries.

The authors of this book take the view that symmetries of dynamical systems used to model applications should be identified and built into any model *a priori*. This approach to dynamics with symmetry is mildly unusual in the field of applied mathematics in that rather than starting with a particular model equation for the dynamics and then performing an analysis of this, the approach first considers properties of the system that are forced simply by the presence of the symmetries. Only when this approach is exhausted does one go on to look at details of the specific system in question. The book illustrates that, while clearly it has its limitations, this approach can be very successful. For example, it often gives an almost complete description of bifurcation properties, up to determination of a handful of real quantities derivable from the system. The same approach can also tell you how many real quantities are to be found and guide you in finding them.

This approach comes into its own in biological systems where, as highlighted recently [6], there is no equivalent of 'Newton's law'. How can one hope to effectively model the interactions between a number of members of a species other than by gross approximation? An example that runs through Chapters 1 and 2 uses the symmetry-based approach of the authors to purposefully ignore this question and just assume that the interacting members of the species are almost identical. They discuss sympatric speciation, or the development of one species into two nearly identical species through evolutionary dynamics in a common environment. Their model uses quasi-static variation of a parameter that results in a steady bifurcation with S_n symmetry for n large. A number of interesting properties of this bifurcation are discussed, such as the fact that the population cannot stably split into two new

structure preserved by the dynamics. These two classes have very different generic behaviour; often different techniques are necessary and different results apply. Much of the impetus for mathematical analysis of dynamical systems, especially in the 19th century, came from conservative dynamics and in particular celestial mechanics. The book reviewed here concentrates for the greater part on dissipative dynamics (where one can find attracting invariant sets) with the exception of Chapter 10, which examines periodic orbits of symmetric Hamiltonian systems.

species that are arbitrarily close to each other; the symmetry forces dynamical consequences, for example that there must be a jump in the measured properties of the phenotype as speciation occurs. In spite of the fact that one cannot clearly interpret even the dimension of the phase space in this problem, the approach is remarkably effective at producing qualitative predictions.

Chapter 3 turns to a totally different problem from biology: the classification of animal gaits (walking patterns) in terms of oscillation patterns of coupled oscillators that belong to a *central pattern generator*, an assemblage of neurons that produces the necessary sequence of muscle activations that result in movement. For example, four-legged animals are known to choose one of a number of possible gaits from the well-known walk, trot and gallop to more obscure gaits such as pronk, loping rack or scuttle. One can use symmetries to classify these gaits in a meaningful way, to gain predictions of the sort of transitions that are possible, and to gain insight into necessary structure within any central pattern generator that drives the locomotion. Again, a lot of the interest in the methods here lies in the fact that one obtains useful predictions even though the central pattern generator is hidden from view.

The subtitle of the book, *From equilibrium to chaos in phase space and physical space*, emphasises that the authors go beyond their previous work [2], [3] both in terms of the dynamical phenomena and in the range of applications considered. These developments include bifurcation theory for systems with permutation symmetry S_n, progress in understanding the bifurcation behaviour of pattern formation systems with noncompact and hidden symmetry groups, and application of symmetry ideas to asymptotic dynamics that is neither periodic nor equilibrium. In the latter case they consider cases of heteroclinic dynamics and chaotic dynamics.

In particular the previous books [2], [3] restrict to systems that have compact symmetry groups; this is because the orthogonal irreducible representations of these groups on likely phase spaces for dynamical systems of interest to application are finite dimensional. This in turn enables one to reduce many bifurcation problems, even for partial differential equations, to low dimensional normal forms. Another aspect that they emphasise is the sometimes subtle connection between the dynamics in phase space and in physical space. Symmetries usually arise in applications owing to their presence in a physical sense. However, it can easily occur that two different systems have the same symmetries of the governing equation but quite different interpretations. This is really a feature of modelling and interpretation rather than of the mathematics *per se*; nonetheless 'mathematical modelling' consists of two words and you ignore the second at your peril! One of the main areas of progress charted in this book has been new results for bifurcations in the presence of noncompact groups and in particular the Euclidean group in Chapters 5 and 6.

Restricting Euclidean group problems to periodic patterns (planforms) allows one to restrict the problem to an invariant subspace with only compact symmetries, and Chapter 5 gives many examples of this approach with applications that include problems such as pattern formation in models of the primary visual cortex and bifurcation in liquid crystal layers. However, one can still make some progress without this restriction. Chapter 6 introduces bifurcation from group orbits to discuss properties of relative equilibria, relative periodic orbits and solutions that bifurcate from them, in particular for continuous groups. This allows one to examine bifurcations from noncompact group orbits of equilibria with compact symmetries. A particular application of this is given for the meandering instability of spiral waves (autowaves) in reaction-diffusion systems on the plane. In some cases, the approach of identifying a model's symmetries and then assuming genericity does not work; this may be because of extra symmetries that are not obvious or that may be induced by boundary conditions. These so-called 'hidden symmetries' are discussed along with examples in Chapter 7.

Regarding the heteroclinic and chaotic dynamics, the results presented are still quite new. Heteroclinic cycles (Chapter 8) are of particular interest in that they can be structurally stable attractors for systems of differential equations with symmetries, but they cannot be so for generic dissipative ODEs. Even worse, robust heteroclinic cycles (cycling chaos) between chaotic invariant sets can also appear in systems with symmetries. The interaction of chaotic attractors with symmetries of the system is discussed in Chapter 9; there are topological obstructions to the existence of chaotic attractors with certain symmetries. Other aspects discussed include the appearance of bifurcations resulting in changes in the symmetry of chaotic attractors and the use of equivariant observables in determining and practically classifying chaotic attractors of systems with symmetries using the intriguingly named 'symmetry detectives'. The main results are essentially conditions on a representation V of the symmetry such that one is able to classify the symmetry of an attractor from generic equivariant observations ϕ taking values in V, for example by considering ergodic averages of ϕ.

The results on heteroclinic and chaotic dynamics are, quite naturally, not as complete as those for equilibria or limit cycles. For example, the book uses a number of different definitions of chaotic attractors using topological attraction (asymptotic stability of an ω-limit set) and measure-theoretic attraction (Sinai-Ruelle-Bowen measures). The definition one uses often results in different answers to the questions that one would like to ask. The final section (Chapter 10) departs from the dissipative dynamics theme to discuss some aspects of Hamiltonian dynamics with symmetries, in particular an equivariant Moser-Weinstein theorem and some examples of application to molecular vibrations.

In the preface, the authors state that they are not aiming for a formal textbook, and in consequence they detail and prove only a selection of the results, referring to the literature for many others (with this in mind, the authors have provided a very extensive bibliography of over 500 items). Rather, the book presents a collection of case studies that serve to illustrate, and indeed that have motivated, recent developments of the theory of dynamical systems with symmetry. Because of this, the book lies somewhere between being a research monograph and a textbook. Nonexperts will find that some sections of the book are self-contained, but others are not and will require reference to other sources (in particular [3]) for a full understanding.

Bearing this in mind, I think that the book will be a great resource, especially for scientists with an application in mind who want to find out what a symmetry-based approach to dynamics can offer them. The applications are of interest in their own right and are not included just for the sake of being examples. Some sections of the book, in particular the introductory chapters and some of the case studies, should be accessible to a first-year graduate student with a basic knowledge of group theory, the qualitative theory of differential equations and an interest in applications. Other sections would be more suited to an advanced graduate student or experienced researcher. This book was awarded the Ferran Sunyer i Balaguer Prize for 2001, and I am sure that it will be a very useful resource not only for researchers specializing in this area but also for those who want to obtain the benefits of using this approach in applications.

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