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Introduction to foliations and Lie groupoids, by I. Moerdijk and J. Mrčun, Cambridge University Press, Cambridge, UK, 2003, x+173 pp., £30.00, ISBN 0-521-83197-0

Foliation theory is multifaceted. We will discuss briefly three of these facets which make up the main subject matter of the book under review.

1. The classical theory

There are many ways in which to describe a (smooth) foliated n-manifold $(M, \mathcal{F})$. By the Frobenius theorem, it is simply an involutive subbundle E of the tangent bundle $T(M)$. If the fibers of E are p-dimensional, the maximal integral manifolds to E are one-to-one immersed submanifolds of M of dimension p, called the leaves. The collection $\mathcal{F}$ of leaves partitions M and, locally, these leaves fit together like a family of parallel p-planes in $\mathbb{R}^n$. The number $q = n - p$ is called the codimension of $\mathcal{F}$ and is somewhat more important than the leaf dimension p. Without explicit reference to the Frobenius theorem, one can *define* a foliation to be such a partition, provided that the tangent spaces to the leaves define a smooth subbundle $E \subseteq T(M)$.

Given such a structure, one constructs a *foliated atlas* $\{U_\alpha, x_\alpha, y_\alpha\}_{\alpha\in\mathfrak{A}}$. Here, the coordinate maps $x_\alpha : U_\alpha \to \mathbb{R}^p$ and $y_\alpha : U_\alpha \to \mathbb{R}^q$ are smooth submersions with images that are open balls in the respective spaces and the level sets of y_α are open p-balls, called plaques, in leaves of $\mathcal{F}$. Each plaque P is coordinatized by $x_\alpha|P$, while y_α can be interpreted as a system of coordinates on a fixed choice of open q-ball $S_\alpha \subset U_\alpha$, transverse to the plaques and meeting each plaque once. The atlas can be chosen so that the local coordinate changes on $U_\alpha \cap U_\beta$ are of the form

$$x_\alpha = x_\alpha(x_\beta, y_\beta)$$
$$y_\alpha = y_\alpha(y_\beta),$$

and so a plaque in U_α overlaps at most one plaque in U_β and vice-versa. The equation $y_\alpha = y_\alpha(y_\beta)$ can be viewed as defining a diffeomorphism $g_{\alpha\beta}$ from an open subset of S_β onto an open subset of S_α. These diffeomorphisms generate a pseudogroup Γ, called the *holonomy pseudogroup* of $\mathcal{F}$, on the open q-manifold $S \subset M$ which is the disjoint union of the transverse balls S_α, $\alpha \in \mathfrak{A}$.

Notice that an atlas with coordinate changes as stipulated makes sense without reference to a foliation, but we still call it a "foliated atlas". One usually adds some additional regularity conditions on the atlas and proves that every regular foliated atlas arises from a unique foliation. From this point of view, the foliation is *defined* by giving a regular foliated atlas. The leaf of $\mathcal{F}$ through a plaque P is recovered as the union of plaques Q that can be reached by a chain of plaques, $P = P_0, P_1, \dots, P_k = Q$, where P_i overlaps P_{i+1}, $0 \le i < k$. One can similarly define foliations of class C^r, $0 \le r \le \infty$, or real analytic foliations ($r = \omega$) by requiring that the foliated atlas be of class C^r. Finally, one can adapt the definition to manifolds M with boundary, each component of ∂M being either a union of leaves of $\mathcal{F}$ (a single leaf if $q = 1$) or transverse to $\mathcal{F}$.

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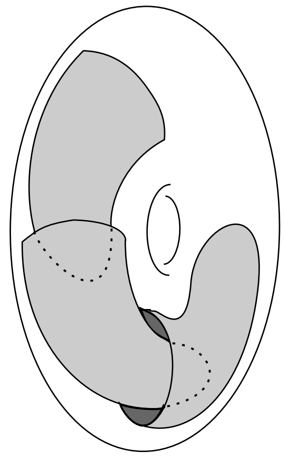

FIGURE 1. The Reeb-foliated solid torus.

Locally a foliation is quite trivial, but globally the structure of the leaves and their asymptotic properties can be quite complicated. This caught the interest of topologists when G. Reeb [15] produced his famous foliation of the 3-sphere. He used the well known decomposition

$$S^3 = S^1 \times D^2 \cup D^2 \times S^1,$$

in which two solid tori are glued together along their common boundary torus by a diffeomorphism that interchanges meridians and longitudes. Each solid torus is foliated as in Figure 1, where the boundary torus T is a leaf and the interior is filled with cup-shaped planes, each winding asymptotically out toward T (snakes repeatedly swallowing their tails). With a little care, this construction can be carried out so that the resulting foliation of S^3 is of class C^∞. Whenever such a foliated solid torus is part of a foliation of a 3-manifold, it is called a *Reeb component*.

A fundamental tool for studying foliations is the *holonomy group* of a leaf. If $L \in \mathcal{F}$, choose $x \in L \cap S$ and consider those elements g of the holonomy pseudogroup Γ that are defined on neighborhoods of x in S and that fix x. The set of germs at x of such $g \in \Gamma$ clearly forms a group under composition. This is the holonomy group $G_x(L)$ of L (at x) and, up to isomorphism, it is independent of the choice of x, of S, and even of the foliated atlas. It can equally well be defined by considering loops on L based at x and defining local diffeomorphisms about x in S by lifting the loops to paths on nearby leaves and taking the first return map. The germs of such maps constitute $G_x(L)$ and, from this point of view, there is a natural group surjection $h : \pi_1(L, x) \to G_x(L)$. Thus, for example, each planar leaf in the Reeb foliation, being simply connected, has trivial holonomy group. One also sees that the toral leaf has $G_x(T) = \mathbb{Z} \oplus \mathbb{Z}$. Notice that each of the two generators of $G_x(T)$ is (the germ of) a contraction on one side of x and the identity on the other side. This holonomy cannot be real analytic, and so the Reeb foliation cannot be of class C^ω.

If L is a compact leaf, $G_x(L)$ characterizes the germ of the foliation at L up to diffeomorphism. This leads to the local Reeb stability theorem [15], which asserts that a compact leaf L with finite holonomy group has a neighborhood made up of compact leaves that cover L. Combined with a theorem of A. Haefliger on the set of compact leaves [6], this gives a global stability theorem for the case in which M is closed and connected and $\mathcal{F}$ has codimension 1 and is transversely orientable (the normal bundle is orientable). If such a foliation has a compact leaf L with $\pi_1(L)$ finite, then all leaves are diffeomorphic to L and are the fibers of a fibration $p : M \to S^1$. A striking generalization of this result, due to W. Thurston [18], replaces the assumption on the fundamental group by the assumption that $H^1(L;\mathbb{R}) = 0$. It is noteworthy that Thurston's result holds only if the foliation is smooth of class at least C^1, while the original Reeb stability result holds for C^0 foliations.

Reeb's foliation of S^3 inspired two other classical results of immense importance. As we noted above, this foliation has 1-sided germinal contracting holonomy, hence cannot be real analytic. Haefliger showed that, for foliations $\mathcal{F}$ of codimension one on a closed manifold with finite fundamental group, 1-sided contracting holonomy is always present and so such foliations cannot be real analytic. His idea was to use a closed, nullhomotopic transversal to $\mathcal{F}$, to place the nullhomotopy in "general" position with respect to $\mathcal{F}$ and apply the Poincaré-Bendixson theorem to the induced singular foliation on D^2. Subsequently, S. P. Novikov [12] used similar techniques to show that, in closed 3-manifolds with finite fundamental group, codimension one foliations must contain a Reeb component. The proofs of these theorems used a general position lemma proven by Haefliger assuming C^2 smoothness, but Novikov's theorem has been extended to C^0 foliations by V. V. Solodov [17].

2. Transverse geometric structures

Throughout this section, the manifold M is assumed to be closed and connected. Part of the discussion extends to open manifolds, but one needs to introduce an extra transverse completeness hypothesis in that case.

As remarked earlier, general foliations can be bewilderingly complex and strong structure theorems are hard to come by. Accordingly, many authors have imposed significant symmetry conditions on foliations and have developed detailed structure theory for such foliations. Of particular importance are foliations carrying a transverse geometric structure. If $E = T(\mathcal{F})$ denotes the subbundle of $T(M)$ tangent to the foliation $\mathcal{F}$, then $Q = T(M)/E$ can be thought of as the normal bundle to $\mathcal{F}$. Thus, just as an infinitesimal geometric structure on M is given by a reduction of the structure group $Gl(n,\mathbb{R})$ of $T(M)$ to a suitable Lie subgroup $G \subset Gl(n,\mathbb{R})$, so one can define a transverse geometric structure as a G-reduction of Q ($G \subset Gl(q,\mathbb{R})$) that, in a suitable sense, is parallel along the leaves of $\mathcal{F}$. The (locally absolute) parallelism can be defined via a certain connection, the Bott basic connection, but rather than go into this, we will reformulate the definition of a transverse structure using the holonomy pseudogroup introduced above. In this language, one gives a geometric structure on the transverse q-manifold S, requiring that the foliated atlas can be chosen so that this structure is invariant under the holonomy pseudogroup Γ.

Of particular note are (transversely) parallelizable foliations and (transversely) Riemannian ones. In the first case there is given a smooth q-frame field ξ on S that is invariant under Γ. Equivalently, there is a normal q-frame field ζ to $\mathcal{F}$ which is parallel relative to the Bott connection. The local submersions $y_\alpha : U_\alpha \to S_\alpha$ project $\zeta|U_\alpha$ exactly to $\xi|S_\alpha$. These foliations are homogeneous in the sense that, given points $x, y \in M$, there is a foliation-preserving diffeomorphism of M carrying x to y.

In the Riemannian case, one is given a Riemannian metric on S and a holonomy pseudogroup Γ consisting of local isometries. In this case, there is a Riemannian metric on Q, invariant under the parallelism along leaves, and this metric extends to a metric on $T(M)$ such that each $y_\alpha : U_\alpha \to S_\alpha$ is a Riemannian submersion. Such metrics were introduced by B. Reinhart [16], who called them "bundle-like".

Also of note are Lie foliations. These are most easily described by an imbedding $S \hookrightarrow G$ relative to which the elements of Γ are locally left translations by elements of the Lie group G. Lie foliations are equivalently given by a nonsingular Maurer-Cartan form. This is a $\mathfrak{g}$-valued 1-form ω on M, where $\mathfrak{g}$ is the Lie algebra of G, $\omega_x : T_x(M) \to \mathfrak{g}$ is surjective, for each $x \in M$, and $d\omega + \frac{1}{2}[\omega, \omega] = 0$. The distribution $\ker(\omega)$ is involutive and the foliation it defines is Lie. Lie foliations were studied by E. Fedida [5].

In what follows we will drop the qualifier "transverse" as no confusion should arise.

Lie foliations are parallelizable and parallelizable foliations are Riemannian, but there are deeper relationships between these notions. If $\mathcal{F}$ is parallelizable, then the closure $\overline{L}$ of each leaf L is a closed submanifold and these submanifolds are the fibers of a smooth fibration $\pi : M \to W$. Furthermore, $\mathcal{F}$ restricts to a foliation of $\overline{L}$, and this is a Lie foliation. By the homogeneity of $\mathcal{F}$, the foliation of M by the fibers of π is also homogeneous and the Lie foliations of the fibers are isomorphic. Thus, the Lie algebra $\mathfrak{g}$ associated to each fiber is an invariant of $\mathcal{F}$.

The structure theorem of the previous paragraph is due to P. Molino [8]. Also, see [9], where his strong structure theorem for Riemannian foliations is also proven. For this theorem, assume that $\mathcal{F}$ is Riemannian. Let $O(M, \mathcal{F})$ be the principle $O(q)$-bundle of orthonormal frames of Q associated to the transverse metric. The locally absolute parallelism of frames along leaves induces a foliation $\widetilde{\mathcal{F}}$ of $O(M, \mathcal{F})$, and Molino proves that $\widetilde{\mathcal{F}}$ is parallelizable. The associated fibration

$$\pi : O(M, \mathcal{F}) \to W$$

is $O(q)$-equivariant relative to an $O(q)$-action on W and the principle $O(q)$-action on $O(M, \mathcal{F})$. The Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω associated to each fiber are thus invariants of $\mathcal{F}$.

3. Lie groupoids and algebroids

The *holonomy groupoid* of a foliation $\mathcal{F}$, also known as the *graph* of $\mathcal{F}$, appears in the work of C. Ehresmann [4], was extensively studied by H. E. Winkelnkemper [21] and used by him to prove a remarkable result about the topology of the leaves of a Riemannian foliation [22]: if M is closed and simply connected, and if $\mathcal{F}$ is a Riemannian foliation, then the leaves have a common universal cover which has 0, 1 or 2 ends. Much deeper applications of this groupoid are due to A. Connes (see, for instance, [2] and [3]), who uses it to construct the C^*-algebra of a foliation (the noncommutative geometry of foliations) and generalizes the Atiyah-Singer index theorem to this context.

A groupoid G is a (small) category with inverses. It consists of two sets, the set G_0 of *objects* and the set G_1 of *morphisms* or arrows between objects. The source and target of each arrow define a pair of maps

$$s, t : G_1 \to G_0,$$

and a pair $(g, h) \in G_1 \times G_1$ is *composable* if $t(h) = s(g)$. The set of composable pairs is denoted by $G_1 \times_{G_0} G_1$, and composition defines a map

$$\mu : G_1 \times_{G_0} G_1 \to G_1,$$

written $\mu(g, h) = gh$. For each $x \in G_0$, there is an identity morphism $1_x : x \to x$, and this defines the unit map

$$1 : G_0 \to G_1.$$

Each arrow is invertible, and this defines the inverse map

$$\iota : G_1 \to G_1.$$

If G_0 and G_1 carry topologies relative to which these structure maps are continuous, G is called a *topological groupoid*. If G_0 and G_1 carry smooth manifold structures such that these structure maps are smooth, G is called a *Lie groupoid*. Because of important examples, one does not demand that the manifold G_1 be Hausdorff.

We turn to the holonomy groupoid. If $\mathcal{F}$ is a foliation of the manifold M, let $G_0 = M$. If $x, y \in M$, the set of arrows $x \to y$ is empty unless x and y lie on a common leaf L of $\mathcal{F}$. If they do lie on L, a path σ on L from x to y defines a holonomy transformation from a small q-transversal through x to one through y. Two such paths are defined to be equivalent if their associated holonomy transformations have the same germ at x. Such an equivalence class is an arrow from x to y. The set G_1 of such arrows is clearly the space of morphisms of a groupoid. With some work, G_1 is given a topology and smooth manifold structure of dimension $2p + q$ (where, as usual, p is the leaf dimension of $\mathcal{F}$ and q is the codimension). The five structure maps are smooth. A related groupoid, the *monodromy groupoid*, is defined by using the relation of homotopy (with fixed endpoints) between paths on a leaf. This too is a Lie groupoid. The holonomy groupoid of a Riemannian foliation is Hausdorff, but for arbitrary foliations this fails. The Reeb foliation of S^3 is an example for which G_1 is not Hausdorff.

An *étale groupoid* is a Lie groupoid in which $\dim G_0 = \dim G_1$. An example of such a groupoid is given by setting $G_0 = \mathbb{R}^q$ and setting $G_1 = \Gamma^q$, the space of germs of locally defined diffeomorphisms in $\mathbb{R}^q$ with the sheaf topology. It was the idea of Haefliger [7] to use this groupoid and a theorem of M. Gromov and A. Phillips (cf. [13]) to classify foliations of open manifolds up to integrable homotopy. By quite different methods, Thurston [19], [20] has obtained analogous results for foliations of arbitrary manifolds.

We have tried to give some motivation for the study of Lie groupoids by sketching their applications to foliation theory. Of course, many other geometric applications exist. And many authors study Lie groupoids for their intrinsic interest as generalizations of Lie groups. In particular, in an effort to establish a close analogy with classical Lie theory, J. Pradines [14] has introduced the notion of a *Lie algebroid* to play the role here of the Lie algebra in the classical theory. Without going into detail, we note that, just as a Lie algebra arises from a Lie group as its infinitesimal approximation, so does each Lie groupoid give rise to a unique associated Lie algebroid. Also, each Lie group is a Lie groupoid (take G_0 to be a singleton), and its associated Lie algebroid is its usual Lie algebra. However, Pradines' expectation that every Lie algebroid would be integrable (associated to a Lie groupoid) proved to be false. A transversely parallelizable foliation of a closed, connected manifold gives rise to a certain Lie algebroid, called the *basic* algebroid, and it was shown by P. Molino and R. Almeida [1] that this algebroid is integrable if and only if the foliation lifted to the universal cover is the foliation by the level sets of a suitable submersion. Examples exist in which this condition fails.

4. The book

The book under review [10] falls into three parts corresponding roughly to the three topics outlined above. The first three chapters cover the classical theory: basic concepts and facts, the stability theorems, Haefliger's theorem and Novikov's theorem. Complete proofs are given of these results. In the case of Thurston's stability theorem, a local version in arbitrary codimension is given from which the global, codimension one version, stated above, follows easily. The proof of this theorem has novel elements, following partly the original proof of Thurston [18] and partly the proof given by the second author in [11].

An important topic in this first part of the book, not alluded to in the above, is that of *orbifolds*. By the local Reeb stability theorem, these arise in foliation theory as the space of leaves of foliations having all leaves compact with finite holonomy.

While Riemannian foliations are introduced in Chapter 2, the detailed treatment of transverse geometric structures is carried out in Chapter 4. This is divided into three sections, the first being devoted to parallelizable foliations. The second section treats principal bundles, connections and transverse principal bundles to a foliation. In the third section, Lie foliations are introduced, and the geometric machinery that has been developed is applied to prove the structure theorems of Molino for parallelizable and Riemannian foliations. It should be remarked that the treatment here is a bit different from what we outlined above. The *projectable* or *basic* vector fields defined by the authors are equivalent to our parallel fields, the Bott basic connection being implicit but not mentioned. The pseudogroup approach to transverse structures is not developed.

Chapters 5 and 6 are the most abstract part of the book. In Chapter 5, Lie groupoids are discussed, the principal examples being the holonomy and monodromy groupoids of a foliation. Also discussed are étale groupoids and proper groupoids and the role of proper groupoids in the theory of orbifolds. An introductory study of Lie algebroids is offered in Chapter 6. Here is discussed the extent to which their relationship to Lie groupoids does and does not parallel that of Lie algebras to Lie groups. The chapter concludes with a proof of the theorem of Almeida and Molino mentioned above.

This book, though relatively short, is packed with interesting information. The student who would like a brief but substantial introduction to foliation theory will find it rewarding, but should be prepared for some hard work. There are many exercises, often challenging, and these illuminate the ideas and sometimes incorporate significant steps of proofs.

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