

*The SIAM 100-Digit Challenge: A study in high-accuracy numerical computing*,  
by Folkmar Bornemann, Dirk Laurie, Stan Wagon, and Jörg Waldvogel, SIAM,  
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In the January 2002 edition of *SIAM News*, Nick Trefethen announced the “\$100, 100-Digit Challenge” [6]. In this note he presented ten easy-to-state but hard-to-solve problems of numerical analysis and challenged readers to find each answer to ten-digit accuracy:

- (1) What is  $\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 x^{-1} \cos(x^{-1} \log x) dx$ ?
- (2) A photon moving at speed 1 in the  $x$ - $y$  plane starts at  $t = 0$  at  $(x, y) = (1/2, 1/10)$  heading due east. Around every integer lattice point  $(i, j)$  in the plane, a circular mirror of radius  $1/3$  has been erected. How far from the origin is the photon at  $t = 10$ ?
- (3) The infinite matrix  $A$  with entries  $a_{11} = 1, a_{12} = 1/2, a_{21} = 1/3, a_{13} = 1/4, a_{22} = 1/5, a_{31} = 1/6$ , etc., is a bounded operator on  $\ell^2$ . What is  $\|A\|$ ?
- (4) What is the global minimum of the function  $\exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y)) - \sin(10(x + y)) + (x^2 + y^2)/4$ ?
- (5) Let  $f(z) = 1/\Gamma(z)$ , where  $\Gamma(z)$  is the gamma function, and let  $p(z)$  be the cubic polynomial that best approximates  $f(z)$  on the unit disk in the supremum norm  $\|\cdot\|_{\infty}$ . What is  $\|f - p\|_{\infty}$ ?
- (6) A flea starts at  $(0, 0)$  on the infinite 2-D integer lattice and executes a biased random walk: At each step it hops north or south with probability  $1/4$ , east with probability  $1/4 + \epsilon$ , and west with probability  $1/4 - \epsilon$ . The probability that the flea returns to  $(0, 0)$  sometime during its wanderings is  $1/2$ . What is  $\epsilon$ ?
- (7) Let  $A$  be the  $20000 \times 20000$  matrix whose entries are zero everywhere except for the primes  $2, 3, 5, 7, \dots, 224737$  along the main diagonal and the number 1 in all the positions  $a_{ij}$  with  $|i - j| = 1, 2, 4, 8, \dots, 16384$ . What is the  $(1, 1)$  entry of  $A^{-1}$ ?
- (8) A square plate  $[-1, 1] \times [-1, 1]$  is at temperature  $u = 0$ . At time  $t = 0$  the temperature is increased to  $u = 5$  along one of the four sides while being held at  $u = 0$  along the other three sides, and heat then flows into the plate according to  $u_t = \Delta u$ . When does the temperature reach  $u = 1$  at the center of the plate?
- (9) The integral  $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)] x^{\alpha} \sin(\alpha/(2 - x)) dx$  depends on the parameter  $\alpha$ . What is the value  $\alpha \in [0, 5]$  at which  $I(\alpha)$  achieves its maximum?
- (10) A particle at the center of a  $10 \times 1$  rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

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Trefethen closed with the enticing comment: “Hint: They’re hard! If anyone gets 50 digits in total, I will be impressed.”

This challenge obviously struck a chord in hundreds of numerical mathematicians worldwide, as 94 teams from 25 nations later submitted entries. Many of these submissions exceeded the target of 50 correct digits; in fact, 20 teams achieved a perfect score of 100 correct digits. Trefethen had offered \$100 for the best submission. Given the overwhelming response, a generous donor (William Browning, founder of Applied Mathematics, Inc.) provided additional funds to provide a \$100 award to each of the 20 winning teams.

Soon after the results were out, four participants, each from a winning team, got together and agreed to write a book about the problems and their solutions. The team is truly international: Bornemann is from Germany, Laurie is from South Africa, Wagon is from the USA, and Waldvogel is from Switzerland.

This book provides some mathematical background for each problem and then shows in detail how each of them can be solved. In fact, multiple solution techniques are mentioned in each case. The book describes how to extend these solutions to much larger problems and much higher numeric precision (hundreds or thousands of digit accuracy). The authors also show how to compute error bounds for the results so that one can say with confidence that one’s results are accurate to the level stated. Numerous numerical software tools are demonstrated in the process, including the commercial products Mathematica, Maple and Matlab. Computer programs that perform many of the algorithms mentioned in the book are provided, both in an appendix to the book and on a website [4].

In the process, the authors take the reader on a wide-ranging tour of modern numerical mathematics, with enough background material so that even readers with little or no training in numerical analysis can follow. Here is a list of just a few of the topics visited: numerical quadrature (i.e., numerical integration), series summation, sequence extrapolation, contour integration, Fourier integrals, high-precision arithmetic, interval arithmetic, symbolic computing, numerical linear algebra, perturbation theory, Euler-Maclaurin summation, global minimization, eigenvalue methods, evolutionary algorithms, matrix preconditioning, random walks, special functions, elliptic functions, Monte-Carlo methods, and numerical differentiation.

It is interesting to note that almost all the major techniques of modern numerical analysis are touched on in this book. Here we see that the field of numerical analysis is much more than a set of lectures on the need to be careful about numerical round-off error; indeed, it encompasses a large and rapidly growing body of clever algorithms and powerful mathematical machinery. As Trefethen himself once observed, “If rounding errors vanished, 95% of numerical analysis would remain” [5].

As mentioned above, the authors show how each problem can be computed to at least 100-digit accuracy and, in several cases, to multi-thousand-digit accuracy. Some may question why anyone cares about such high levels of precision when hardly any quantities in the physical world are known to an accuracy beyond about 12 decimal digits.

As it turns out, recent work in the emerging discipline of experimental mathematics has provided a compelling motivation for such computations, since very high precision is often required when applying integer relation algorithms to discover new mathematical identities. An integer relation algorithm is an algorithm that, given  $n$  real numbers  $(x_i, 1 \leq i \leq n)$ , produces  $n$  integers, not all zero, such

that  $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ . Probably the best known application of an integer relation algorithm is the discovery in 1995 of the “BBP” formula for  $\pi$ , which permits binary or base-16 digits of  $\pi$  to be computed starting at an arbitrary position [1]. The BBP formula was found by a computer program implementing the PSLQ integer relation algorithm using a numerical precision of approximately 200 digits.

Several recent results in experimental mathematics are even better examples of applications of the computational techniques described in the 100-digit challenge book. As one example, recently the present author, together with Jonathan Borwein of Dalhousie University, discovered that

$$(1) \quad \frac{2}{\sqrt{3}} \int_0^1 \frac{\log^6(x) \arctan[x\sqrt{3}/(x-2)]}{x+1} dx \\ = \frac{1}{81648} [-229635L_3(8) + 29852550L_3(7) \log 3 - 1632960L_3(6)\pi^2 \\ + 27760320L_3(5)\zeta(3) - 275184L_3(4)\pi^4 + 36288000L_3(3)\zeta(5) \\ - 30008L_3(2)\pi^6 - 57030120L_3(1)\zeta(7)],$$

where  $L_3(s) = \sum_{n=1}^{\infty} [1/(3n-2)^s - 1/(3n-1)^s]$ , by numerically computing each of the basic terms in this formula to 1000-digit accuracy and applying the PSLQ algorithm. A proof was subsequently found. A second example is the following:

$$(2) \quad \frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt \stackrel{?}{=} L_{-7}(2)$$

where

$$L_{-7}(s) = \sum_{n=0}^{\infty} \left[ \frac{1}{(7n+1)^s} + \frac{1}{(7n+2)^s} - \frac{1}{(7n+3)^s} \right. \\ \left. + \frac{1}{(7n+4)^s} - \frac{1}{(7n+5)^s} - \frac{1}{(7n+6)^s} \right].$$

In this case the “identity” (2) has been verified to over 20,000 decimal digit accuracy, but a proof is not yet known [2, pp. 90–91]. A related experimental result, verified to 1000 digit accuracy but so far unproven, is

$$0 \stackrel{?}{=} -2J_2 - 2J_3 - 2J_4 + 2J_{10} + 2J_{11} + 3J_{12} + 3J_{13} + J_{14} - J_{15} \\ - J_{16} - J_{17} - J_{18} - J_{19} + J_{20} + J_{21} - J_{22} - J_{23} + 2J_{25}$$

where  $J_n$  is the integral in (2), with limits  $n\pi/60$  and  $(n+1)\pi/60$ . Numerous other examples of this methodology in action can be found in [2],[3].

Perhaps the most significant contribution of this book, however, is how it demonstrates that a set of well-chosen problems can serve as basis for introducing a reader to an entire field. Indeed, this book could be used as a text for an advanced undergraduate or graduate course in numerical analysis. Students, in reading this book and experimenting with the techniques presented, can readily see the motivation for each of the topics that they study. Furthermore, they can gain first-hand knowledge of the relative power of various techniques. It would be very interesting to see how this model of exposition works for other topics in mathematics.

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