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Introduction to Möbius differential geometry, by Udo Hertrich-Jeromin, London Mathematical Society Lecture Notes Series, vol. 300, Cambridge University Press, Cambridge, UK, 2003, xi+413 pp., US$50.00, ISBN 0-521-53569-7

Möbius differential geometry is a classical subject that was extensively developed in the nineteenth and early twentieth centuries, culminating with the publication of Blaschke's *Vorlesungen über Differentialgeometrie III: Differentialgeometrie der Kreise und Kugeln* [2] in 1929. Research in the field was less intense over the next fifty years, but a resurgence of activity began about 1980, leading to some of the most interesting developments in submanifold theory over the past 25 years. The book under review lays the groundwork for both the classical and modern developments in the theory, leading the reader to the frontiers of research on several topics.

Möbius (conformal) differential geometry grew out of the classical study of curves and surfaces in 3-dimensional Euclidean space $\mathbf{R}^3$. In Möbius geometry, there is measurement of angles but no measurement of lengths, and one searches for geometric quantities that are invariant under the group of conformal transformations of the ambient space. It is possible to study Möbius geometry of surfaces in $\mathbf{R}^3$ by placing the usual Euclidean metric on $\mathbf{R}^3$ and trying to find Euclidean geometric quantities which are invariant under the group of conformal transformations of $\mathbf{R}^3$. However, this is often cumbersome, and it is not the classical approach of Blaschke's book nor of the book under review. We begin by describing this classical approach in some detail and later will briefly describe two modern formulations of the theory which are also developed in this book.

The classical model of Möbius geometry is situated in the context of projective geometry. Since $\mathbf{R}^n$ is conformally equivalent via stereographic projection to $S^n - \{p\}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$ and $p \in S^n$ is the pole of the stereographic projection, one can use S^n instead of $\mathbf{R}^n$ as the base space in the study of conformal geometry. This is often simpler, since S^n is compact, and there is no need to consider an improper point ∞ nor to distinguish between hyperspheres and hyperplanes.

The construction of the classical model begins by embedding $\mathbf{R}^{n+1}$ as an affine subspace of real projective space $\mathbf{R}P^{n+1}$ as follows. Consider the Lorentz space $\mathbf{R}^{n+2}_1$ with Lorentz inner product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \ldots + x_{n+1}y_{n+1}, \tag{1}$$

for $x = (x_0, x_1, \ldots, x_{n+1}), y = (y_0, y_1, \ldots, y_{n+1})$ in $\mathbf{R}^{n+2}_1$. A vector $x \in \mathbf{R}^{n+2}_1$ is said to be *spacelike*, *timelike* or *lightlike*, respectively, depending on whether $\langle x, x\rangle$ is positive, negative or zero. The set L^{n+1} of all lightlike vectors in $\mathbf{R}^{n+2}_1$, given by the equation

$$x_0^2 = x_1^2 + \ldots + x_{n+1}^2, \tag{2}$$

forms a cone of revolution called the *light cone*. Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone". Now consider the real projective space $\mathbf{R}P^{n+1} = \mathbf{R}^{n+2}_1/\sim$, where $x \sim y$ if x is a non-zero scalar multiple of y, and

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let $[x]$ denote the projective equivalence class of the vector $x \in \mathbf{R}^{n+2}_1$. In $\mathbf{R}P^{n+1}$, one can embed $\mathbf{R}^{n+1}$ as the affine space of points $[(x_0, x_1, \ldots, x_{n+1})]$ with $x_0 \neq 0$. Each such point has a unique representative of the form $[(1, y_1, \ldots, y_{n+1})] = [(1, y)]$, with $y_i = x_i/x_0$, for $1 \leq i \leq n+1$, and $y \in \mathbf{R}^{n+1}$. Note that for $x = (1, y)$, one has

$$\langle x, x\rangle = \langle (1, y), (1, y)\rangle = -1 + y \cdot y, \tag{3}$$

where $y \cdot y$ is the usual Euclidean inner product on $\mathbf{R}^{n+1}$. Thus $\langle x, x\rangle = 0$ if and only if $y \cdot y = 1$, that is, $y \in S^n$. So the set $L^{n+1}/\sim$ of projective classes of lightlike points in $\mathbf{R}^{n+2}_1$ is diffeomorphic to the unit sphere S^n in $\mathbf{R}^{n+1}$.

Objects of fundamental importance in Möbius geometry are the hyperspheres in S^n. Consider the hypersphere S in S^n with center p and spherical radius ρ, $0 < \rho < \pi$. The hypersphere S is the intersection of S^n with the hyperplane in $\mathbf{R}^{n+1}$ given by the equation

$$p \cdot y = \cos\rho, \quad 0 < \rho < \pi. \tag{4}$$

If we let $[(1, y)]$ be a point in the affine space, then (4) can be rewritten as

$$\langle (1, y), (\cos\rho, p)\rangle = 0. \tag{5}$$

Thus a point $y \in S^n$ is in the hyperplane given by (4) if and only if $[(1, y)]$ lies in the polar hyperplane in $\mathbf{R}P^{n+1}$ of the point

$$[\xi] = [(\cos\rho, p)]. \tag{6}$$

Note that

$$\langle \xi, \xi\rangle = -\cos^2\rho + p \cdot p = 1 - \cos^2\rho = \sin^2\rho > 0, \tag{7}$$

for $0 < \rho < \pi$, so that ξ is a spacelike vector in $\mathbf{R}^{n+2}_1$. Thus, the hypersphere S given by equation (4) can be identified with the spacelike point $[\xi]$ in $\mathbf{R}P^{n+1}$. So the space of hyperspheres in S^n is diffeomorphic to the "outer space" of spacelike points in projective space,

$$\mathbf{R}P^{n+1}_O = \{[\xi] | \langle \xi, \xi\rangle > 0\}. \tag{8}$$

Note that $[\xi] = [\zeta]$, where $\zeta = \xi / \pm\sin\rho$ and $\langle \zeta, \zeta\rangle = 1$. Thus, the *Lorentz sphere*,

$$S^{n+1}_1 = \{\zeta \in \mathbf{R}^{n+2}_1 | \langle \zeta, \zeta\rangle = 1\}, \tag{9}$$

is a double cover of $\mathbf{R}P^{n+1}_O$. Note that one can distinguish the two possible orientations of the sphere S by assigning a plus or minus sign to the radius ρ. This leads to Lie's geometry of oriented spheres, which is actually the point of view of Blaschke's book (see also [4] and [10]), in which Möbius geometry is considered as a subgeometry of Lie sphere geometry.

By a direct calculation, one can show that the two hyperspheres S_1 and S_2 corresponding to the projective points $[\xi_1]$ and $[\xi_2]$, respectively, intersect orthogonally if and only if

$$\langle \xi_1, \xi_2\rangle = 0, \tag{10}$$

and one can determine the angle of intersection between any two spheres by a similar formula involving the Lorentz inner product.

By definition, a *Möbius transformation* is a diffeomorphism $\mu : S^n \to S^n$ that maps hyperspheres to hyperspheres. Thus, a Möbius transformation induces a map from the space of spheres $\mathbf{R}P^{n+1}_O$ to itself, and using the Fundamental Theorem of Projective Geometry, one can show that this transformation naturally extends to a unique projective transformation on $\mathbf{R}P^{n+1}$. In this way, a Möbius transformation can be identified with a projective transformation on $\mathbf{R}P^{n+1}$ that maps the space $S^n = L^{n+1}/\sim$ onto itself. One can then prove that such a projective transformation must be induced by an orthogonal transformation $A \in O_1(n+2)$ of $\mathbf{R}_1^{n+2}$. Such a projective transformation must preserve the polarity condition (10) and therefore preserve angles between hyperspheres, so it corresponds to a conformal transformation of S^n. Conversely, Liouville's Theorem states that if $\phi : U \to V$ is a conformal diffeomorphism between connected open subsets U and V in $S^n, n \geq 3$, then there is a unique Möbius transformation μ such that ϕ is the restriction of μ to U.

We now discuss the study of submanifolds in Möbius geometry. Basically, submanifolds are considered as envelopes of differentiable families of spheres. To make this precise, one defines a *sphere congruence* to be a differentiable m-parameter family of spheres $S : M^m \to \mathbf{R}P_O^{n+1}$. A differentiable map $f : M^m \to S^n$ is called an *envelope* of a sphere congruence S if for all $p \in M^m$,

$$f(p) \in S(p) \text{ and } T_{f(p)}f(M^m) \subset T_{f(p)}S(p), \tag{11}$$

where $T_{f(p)}f(M^m)$ and $T_{f(p)}S(p)$ denote the tangent subspaces to $f(M^m)$ and $S(p)$, respectively, at the point $f(p)$. Neither f nor S is required to be an immersion, although such conditions are often assumed. One can formulate the enveloping conditions (11) in terms of the Lorentz inner product as follows. First, one knows that $\langle f, f\rangle = 0$, since f maps into S^n. Further, recalling that the Lorentz sphere S_1^{n+1} is a double cover of $\mathbf{R}P_O^{n+1}$, one can locally choose homogeneous coordinates for the sphere congruence S so that $\langle S, S\rangle = 1$. Then one can easily show that in this situation, the two conditions in (11) become, respectively,

$$\langle f, S\rangle = 0 \text{ and } \langle df, S\rangle = 0. \tag{12}$$

Furthermore, differentiating the condition $\langle f, S\rangle = 0$ and using $\langle df, S\rangle = 0$, one obtains

$$\langle f, dS\rangle = 0. \tag{13}$$

Thus, one can interpret this in two different ways. One can consider S as a spacelike unit normal field for the mapping f, or one can consider f as a lightlike normal field for the mapping S. Both points of view are important and are used to prove important results in the book. A pair of maps $f : M^m \to L^{n+1}$ and $S : M^m \to S_1^{n+1}$ that satisfy (12) is called a *strip*, and strips are the basic setup for local differential geometry in Möbius geometry.

An important concept in Möbius differential geometry is the notion of the central sphere congruence of an immersion. Specifically, given an immersion $f : M^{n-1} \to S^n$, there is precisely one sphere congruence $Z : M^{n-1} \to S_1^{n+1}$ such that the corresponding shape operator (Weingarten tensor) A_Z determined by Z as a normal field of f has trace zero. This sphere congruence is called the *central sphere congruence* of f in Blaschke's book. In recent times, Z is often referred to as the *conformal Gauss map* of f, a description given in a 1982 paper [3] of Bryant. Geometrically, the central sphere $Z(p)$ is the sphere tangent to $f(M^{n-1})$ at $f(p)$ which has the same mean curvature $H(p)$ as the hypersurface $f(M^{n-1})$ at the point $p \in M^{n-1}$. Thus, $Z(p)$ is sometimes referred to as the *mean curvature sphere* at p.

The most outstanding problem in Möbius differential geometry is the Willmore Conjecture. This conjecture is most naturally formulated in terms of surfaces in $\mathbf{R}^3$ rather than S^3. Let $f : M^2 \to \mathbf{R}^3$ be a compact surface immersed in $\mathbf{R}^3$. Let k_1

and k_2 denote the principal curvatures of f, and let $H = (k_1 + k_2)/2$ and $K = k_1 k_2$ denote the mean and Gauss curvatures of f, respectively. In 1965 Willmore [14] proposed the study of the functional

$$\tau(f) = \int_{M^2} H^2 \, dA, \tag{14}$$

where dA is the area form on M^2 induced by the immersion f. Using the central sphere congruence Z mentioned above, several authors including Fubini [6], Thomsen [12] and White [13] have proven that the two-form $H^2 - K\, dA$ is Möbius invariant. Thus, the so-called *Willmore functional*,

$$W(f) = \int_{M^2} H^2 - K \, dA, \tag{15}$$

is Möbius invariant. Since the Gauss-Bonnet Theorem states that

$$\int_{M^2} K \, dA = 2\pi\chi(M^2), \tag{16}$$

where $\chi(M^2)$ is the Euler characteristic of M^2, we have

$$W(f) = \int_{M^2} H^2 - K \, dA = \tau(f) - 2\pi\chi(M^2), \tag{17}$$

and so $\tau(f) = W(f) + 2\pi\chi(M^2)$ is also Möbius invariant. Note that

$$H^2 - K = \frac{1}{4}(k_1 - k_2)^2, \tag{18}$$

so the Willmore functional has the property that its integrand is always nonnegative, and it vanishes precisely at umbilic points where $k_1 = k_2$.

Willmore noted that $\tau(f) \geq 4\pi$ for any immersion $f : M^2 \to \mathbf{R}^3$ and that equality holds if and only if $M^2 = S^2$ and f embeds M^2 as a round (totally umbilic) sphere in $\mathbf{R}^3$. He then turned his attention to immersions of the torus T^2 into $\mathbf{R}^3$, in particular "anchor rings" obtained by rotating a circle about an axis in $\mathbf{R}^3$. Willmore proved that $\tau(f) \geq 2\pi^2$ for such anchor rings, and equality holds precisely when the ratio of the radius to the distance from the axis to center of the circle is $1/\sqrt{2}$. He then formulated the *Willmore Conjecture*, which states that $\tau(f) \geq 2\pi^2$ for all immersions $f : T^2 \to \mathbf{R}^3$. As noted above, White [13] pointed out that $\tau(f)$ is Möbius invariant, and thus the value $\tau(f) = 2\pi^2$ is also attained for those cyclides of Dupin which are Möbius equivalent to an anchor ring with $\tau(f) = 2\pi^2$. An alternative description of this family of cyclides of Dupin is that it consists of all surfaces that are Möbius equivalent to the image under stereographic projection of a minimal Clifford torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3$.

There has been extensive and deep research on the Willmore Conjecture which still remains open. A recent paper by Willmore [15] is an excellent survey of results on this problem. A few of the key contributions are the following. In 1982, Li and Yau [9] defined the notion of the conformal volume $V_c(M)$ of a compact Riemannian manifold M with fixed conformal structure. Li and Yau showed that for a conformal immersion $f : M^2 \to \mathbf{R}^3$, the inequality $\tau(f) \geq V_c(M^2)$ holds and that $V_c(M^2) \geq 2\pi^2$ for a torus with a conformal structure near to that of a minimal Clifford torus. In 1982, Bryant [3] gave a Möbius geometric treatment of the variational problem for the Willmore functional among other results. In 1989, Kusner [8] considered the problem of minimizing $\tau(f)$ for immersed surfaces of arbitrary genus, including non-orientable surfaces. In particular, he found an upper bound for the infimum of

$\tau(f)$ among all compact immersed surfaces of each given topological type. Several other authors have given generalizations of the Willmore problem to submanifolds of higher dimension and codimension. In the book, the author provides a proof [7] of the Willmore Conjecture for umbilic-free channel (canal) tori, that is, surfaces that are envelopes of a 1-parameter family of spheres.

Aside from the classical projective model for Möbius differential geometry, the book has extensive chapters on two other models which have arisen in recent research. These are the quaternionic model and the Clifford algebra model. The quaternionic model is a generalization to higher dimensions of the description of Möbius transformations of $S^2 \simeq \mathbf{C}P^1 \simeq \mathbf{C} \cup \{\infty\}$ as fractional linear transformations. For dimensions 3 and 4, this can be done using quaternions. The key point here is that points and hyperspheres in the conformal 4-sphere $\mathbf{H}P^1 \simeq \mathbf{H} \cup \{\infty\}$, where $\mathbf{H}$ is the skew-field of quarternions, can be identified with lightlike and spacelike directions in the space of quaternionic Hermitian forms on $\mathbf{H}^2$, respectively. In a lengthy recent paper [5], Ferus, Leschke, Pedit and Pinkall used this approach to make substantial progress in the global Möbius differential geometry of surfaces.

The Clifford algebra approach has its roots in the early papers of Grassmann and Clifford on the Clifford algebra $\mathcal{A}\mathbf{R}_1^{n+2}$ of the $(n+2)$-dimensional space of homogeneous coordinates on $\mathbf{R}P^{n+1}$ equipped with the Lorentz inner product. An important feature of this model is that the orientation preserving Möbius transformations are described by the elements of the spin group $Spin(\mathbf{R}_1^{n+2}) \subset \mathcal{A}\mathbf{R}_1^{n+2}$. A second feature of this model is that the space of all spheres of any codimension is rather easily described.

The book provides substantial treatments of several classical topics which are once again the subject of active research, including: conformally flat hypersurfaces, isothermic surfaces, Darboux transforms, Ribaucour transformations and Ribaucour pairs of orthogonal systems, Guichard nets and Christoffel transforms, to name only a few.

The book is a welcome addition to the literature, as there are only a few books on the subject. Aside from the book of Blaschke [2] mentioned earlier, another classical text is the book of Takasu [11] published in 1938 and also written in German. A modern treatment of conformal differential geometry is the book of Akivis and Goldberg [1] published in 1996. This is a good book with an extensive bibliography which presents the classical theory as well as generalizations to higher dimensions and codimensions. It differs substantially from the book under review in terms of the topics covered and the techniques employed, however.

The book under review is significant for both its presentation of the classical approach to Möbius differential geometry and its exposition of recent developments in the field. It is carefully written with detailed references, extensive historical notes and helpful comments at the beginning of each chapter. It is an outstanding introduction to a classical field which has taken on new life over the past quarter century, and it is highly recommended by the reviewer.

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