BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, Pages 279–284
S 0273-0979(06)01087-1
Article electronically published on February 16, 2006

Positivity in algebraic geometry. I–II, by Robert Lazarsfeld, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer-Verlag, Berlin, 2004, (I) xviii+387 pp., ISBN 3-540-22533-1; (II) xviii+385 pp., ISBN 3-540-22534-X; US$129.00 (hardcover), US$34.95 (softcover)

As we all learned in our early encounters with $i = \sqrt{-1}$, it makes no sense to talk about *positive* complex numbers. Thus it is all the more surprising that notions of positivity play a key role in complex geometry. Here positivity should be thought of in a broad sense as the presence of a natural order or partial order on a collection of objects. Let us start with three examples illustrating this claim, contrasting the appearance of positivity in complex geometry with its absence in real geometry.

Example 1. Let M^n be a compact topological manifold. Which elements in $H_k(M, \mathbb{Z})$ can be realized as homology classes of k-dimensional submanifolds? The typical answer, as noted by Thom, is that every class can be realized this way (cf. [Sul04]).

By contrast, let X^n be a compact complex submanifold of some $\mathbb{CP}^N$. The homology classes of k-dimensional complex submanifolds span a convex cone, denoted by $NE_k(X) \subset H_{2k}(X, \mathbb{R})$, which is a cone in the naive sense of the word. (That is, it contains no straight lines.) Thus we can distinguish "positive" homology classes (the ones in $NE_k(X)$) from "negative" homology classes (the ones in $-NE_k(X)$).

Example 2. Again let M^n be a compact topological manifold and $E \to M$ a real vector bundle. Any inner product on E shows that E is isomorphic to its dual E^*. Furthermore, E is always spanned by global sections; equivalently, it can be written as the quotient of a trivial bundle $\mathbb{R}^m \times M \to M$.

The situation is completely different for holomorphic bundles on compact complex manifolds. Here E is usually quite different from its dual. Consider for instance the case when E is spanned by global sections. Any global section of E^* gives global sections of the quotient $E \otimes E^* \to \mathbb{C} \times M$, and by the maximum principle any global section of $\mathbb{C} \times M \to M$ (that is, a holomorphic map $M \to \mathbb{C}$) is constant. Thus eventually we obtain that E is isomorphic (as a holomorphic bundle) to its dual only if E is the trivial bundle.

In general, we can distinguish "positive" bundles (for instance, those with plenty of holomorphic sections) from "negative" bundles (whose dual has plenty of holomorphic sections).

Example 3. While compact topological surfaces neatly fall into three groups according to the existence of metrics of positive/flat/negative curvature, higher-dimensional real manifolds cannot be similarly divided. Most manifolds do not have metrics whose sectional curvature has a fixed sign. Considering the weaker Ricci curvature, one gets that every 3–manifold admits a metric with negative Ricci curvature [GY86], making the distinction between positively and negatively curved manifolds somewhat meaningless.

2000 *Mathematics Subject Classification.* Primary 14-02, 14C20, 14F05, 14F17.

Complex manifolds by and large preserve the trichotomy of surface theory. First of all, if a complex manifold has a Kähler metric whose Ricci curvature is positive/flat/negative, then it cannot have any other Kähler metric whose curvature has a different sign. Existence is a thornier issue, but the minimal model program (or Mori's program) roughly asserts that every complex manifold can be assembled from positive/flat/negative pieces using some complex surgery operations. (See [KM98] for an introduction.)

Lazarsfeld's book gives a thorough discussion of the ideas, methods and results that originate with these three basic examples. More precisely, the author concentrates on the first two examples and on the general features and properties of positivity. The aim is to explain the global picture with general theorems, and use the special cases as interesting examples. For instance, much work has been devoted to deciding which homology classes can be realized by submanifolds and to computing the exact dimension of the space of global sections of vector bundles, but these appear only as occasional detours in the book.

We see later how the third example above would fit into this approach and why it should be treated quite differently.

In order to understand positivity in algebraic geometry, one has to start with line bundles and the maps given by them. Let $L \to X$ be a complex holomorphic line bundle on a compact complex manifold and $s_0, \dots, s_n : X \to L$ global sections. For any $x \in X$ the sections give $n+1$ numbers

$$s_0(x), \dots, s_n(x) \in L_x \cong \mathbb{C}.$$

The isomorphism $L_x \cong \mathbb{C}$ is an isomorphism of 1–dimensional vector spaces, so is unique up to multiplication by a nonzero complex number. Thus L and the sections $s_0, \dots, s_n$ define a map not to $\mathbb{C}^{n+1}$ but only to projective n-space $X \dashrightarrow \mathbb{CP}^n$. I use a dashed arrow to indicate that the map is not defined at the points where all the s_i vanish. A line bundle L is called *very ample* if we can choose sections $s_0, \dots, s_n$ (for some n) such that the corresponding map $X \dashrightarrow \mathbb{CP}^n$ is an everywhere defined embedding. We like to think of very ample line bundles as the "most positive" ones.

Algebraic geometers love very ample line bundles, since they provide a way to rigidify a complex manifold. In many cases one can prove that a given manifold X has a *unique* embedding into some $\mathbb{CP}^n$, up to a linear change of coordinates. Usually one can introduce in this way distinguished global coordinates and make sense of elementary geometric notions like collinearity on a complex manifold. While global coordinates and collinearity are rarely used directly, the resulting rigidity of complex geometry is a salient feature of the field.

Thus the first step in any study of positivity is to understand very ample line bundles and their relatives. Accordingly, part one of the book starts to answer the following basic questions:

– How can one tell which line bundles are very ample?

– Which line bundles are close to being very ample?

– How can one create very ample line bundles?

– How can one use a very ample line bundle to understand the structure of the underlying manifold?

Before turning to the book under review in detail, I should disclose that I am not a disinterested commentator. I have seen and read several of the preliminary versions and was very happy when Lazarsfeld chose to publish it in the Ergebnisse series, of which I am an editor. So, gentle reader, do not rely too much on my appraisal. Get a copy of the book and read it yourself.

The treatment is divided into three parts. The first and longest part (Chs. 1–5) makes up volume I [Laz04a]. Part 2 (Chs. 6–8) and part 3 (Chs. 9–11) together fill volume II [Laz04b]. This division provides two volumes of identical thickness, but does not make much mathematical sense.

Parts one and two treat the traditional theory of positivity in algebraic geometry, most of whose basic definitions and results have been established for at least 20 years. Modern developments have led to some changes in proofs and emphasis, but the basic framework has been stable, and it is ready for a monograph that serves as the definitive summary of the field for decades to come. I believe that Lazarsfeld's volumes give this treatment. At the same time, the presentation is easy to follow, thus it can also be used in a second course of algebraic geometry. The theory is laid out clearly, emphasizing the main ideas and theorems, and a huge number of examples also show the many ramifications, applications and special cases that have been studied or are worth exploring further.

By contrast, part 3 is the first detailed treatment of a rapidly developing new area, thus it is likely to become dated sooner. For now it is so fresh that Sec.11.4.C describes the dual of the cone of divisors $NE_{\dim X-1}(X)$, whose original proof (by Boucksom, Demailly, Paun and Peternell) has not yet appeared in print.

In order to obtain a more detailed overview of positivity in algebraic geometry, we can as well follow the presentation of [Laz04a, Laz04b].

Chapter 1 deals with positivity for line bundles and its relation to algebraic curves (= compact Riemann surfaces) on a given complex manifold. These two are linked by a beautiful duality which is one of the starting points of the theory. The first Chern classes of very ample line bundles span a cone $\operatorname{Amp}(X) \subset H^2(X, \mathbb{R})$, and the homology classes of curves span the cone of curves $NE_1(X) \subset H_2(X, \mathbb{R})$. Cohomology is the dual of homology, and under this pairing the cone of very ample line bundles $\operatorname{Amp}(X)$ is the interior of the dual of the cone of curves $NE_1(X)$ (Kleiman's theorem).

The cone $NE_1(X)$ and its dual $\operatorname{Amp}(X)$ are very interesting invariants of X, and one can learn much about X by studying these cones, especially their boundaries. A line bundle whose Chern class is in the closure of $\operatorname{Amp}(X)$ is called *nef*. (While historically this is an abbreviation of "numerically effective", the latter is a quite misleading name.) Many of the interesting – and for applications crucial – questions concern the difference between nef and very ample line bundles.

The main topic of Chapter 2 is the (lack of) Zariski decomposition, whose aim, roughly speaking, was to write any line bundle in terms of a very ample line bundle and another part that can never contribute to ampleness. By now this theory consists entirely of counterexamples, so its main purpose is to get the expectations right. There are, however, many instances where this idea works very well, so decades of counterexamples could not shake my hope that some day there would be a nice, though limited, theory.

The study of very ample line bundles is essentially equivalent to the theory of hyperplane sections, first brought to prominence by Lefschetz. He proved that if $X \subset \mathbb{CP}^N$ is a compact, complex submanifold and $X \cap H \hookrightarrow X$ a smooth hyperplane section, then the induced maps $H_i(X \cap H, \mathbb{Z}) \to H_i(X, \mathbb{Z})$ are isomorphisms for $i < \dim(X \cap H)$. (Here $\dim X$ is the complex dimension, so the isomorphisms hold below one half of the real dimension.) Poincaré duality then tells us that the only "new" homology of $X \cap H$ appears in the middle dimension. Many similar results relating X to its sufficiently positive submanifolds are also discussed in Chapter 3.

If I had to pick a result from positivity which had the greatest influence in algebraic geometry, it would be the Kodaira vanishing theorem, discussed in Chapter 4. (Though no doubt, many would pick the theorem of Lefschetz mentioned above.) The theorem says that if $L \to X^n$ is a positive line bundle, then the cohomology groups of the dual line bundle $H^i(X, L^*)$ are zero for $i < n$. Without going into detail, there are two types of situations where cohomology vanishing theorems are very useful:

– local constructions can be globalized if some cohomology group is zero, and

– lifting information from a lower-dimensional subvariety also requires cohomology vanishing.

While the Kodaira vanishing theorem shows that many cohomology groups are zero, in application one almost always needs some new twist. Several of these will be discussed in part 3.

The notion of curvature in differential geometry is a completely local property which can be used to measure the very global concept of positivity. There is no similar way to make positivity local in algebraic geometry, but the notion of *Seshadri constants*, examined in Chapter 5, gives a way to measure the local positivity of a line bundle on a complex manifold X in terms of its behavior on algebraic curves in X. These results are relatively new, and even some basic questions are open.

Part 2 extends the theory of positive line bundles to positive vector bundles. The precise definition of positive vector bundles is still unsettled, since it is not known whether or not the notion favored by algebraic geometers (ampleness) is the same as the one coming from differential geometry (Griffiths positivity). For many applications this does not matter, however, and we are treated to a nice presentation of the main implications of positivity:

– surprisingly many expressions involving Chern classes of ample vector bundles are positive (Griffiths, Fulton and Lazarsfeld) and

– a map between vector bundles $u : E \to F$ drops rank on a connected nonempty subset if the naive dimension count suggests that it should and $F \otimes E^*$ is ample (Fulton and Lazarsfeld).

Both of these have numerous applications, many of which are explained in theorems and exercises.

This completes the discussion of the traditional theory of positivity in algebraic geometry, and there are two natural directions to pursue from here. The first one is positivity theory for the *canonical* line bundle. Given a complex manifold X, the determinant of the holomorphic tangent bundle $\det T_X$ is a very interesting line bundle. For technical reasons it is better to work with its dual K_X, which is the *canonical* line bundle of X. As we understand today, almost all of the useful general information about a complex manifold is concentrated in its canonical line bundle.

It turns out that the canonical line bundle behaves much better in positivity questions than an arbitrary line bundle. The difference is quite shocking and in many ways still poorly understood. Following this thread leads to the classification of algebraic surfaces and to its higher-dimensional incarnation, Mori's program. While this would have been a logical direction to pursue, it turns out to be technically quite different. Also, several recent monographs already address this subject [KM98, Deb01, Mat02].

Instead, part 3 moves us into the rapidly developing theory of multiplier ideals, which may also be called the theory of "approximate positivity".

This idea appeared simultaneously in algebraic geometry and in complex differential geometry, the key observation being that some consequences of positivity (chiefly the Kodaira vanishing theorem) hold for line bundles that are "close to being positive". This vague statement is not surprising at all. It is, however, remarkable that in applications many line bundles turn out to be close to being positive, and therefore these techniques are useful in quite unexpected areas. The abundance of applications explains the continuing flowering of this field. Since these are relatively new ideas, let us digress a little to give the definitions.

On the algebraic geometry side, the Kodaira vanishing theorem was generalized by Kawamata and Viehweg to line bundles $L \to X$ with the following very artificial sounding property:

– one can write $c_1(L) = \gamma c_1(A) + \sum \alpha_i [D_i]$, where A is very ample, $\gamma > 0$, the $D_i \subset X$ are smooth hypersurfaces intersecting transversally whose homology class $[D_i]$ can be thought of as a second cohomology class, and $1 > \alpha_i > 0$.

In practice it is difficult to ensure that the D_i be smooth and intersect transversally, but resolution of singularities allows one to achieve this after some blowups.

Differential geometers noticed that one can frequently work with *singular* metrics on vector bundles. Normally a metric on a vector bundle is locally given by a symmetric matrix $\big(h_{ij}(\mathbf{z})\big)$ using a local frame $\{e_i\}$ where the h_{ij} are C^∞ functions. If we allow the functions h_{ij} to blow up, we get singular metrics. The relevant observation, as emphasized by Demailly, is that as long as the h_{ij} are L^2 and second derivatives make sense as distributions, many of the consequences of the traditional positivity theory still hold.

In both cases, one can interpret the results by saying that the positivity properties of L are concentrated in a particular subsheaf of L. By all rights, it should be called the *multiplier subsheaf* of L, but subsheaves of a line bundle can be identified with ideal sheaves, and so they are called *multiplier ideals*. (Roughly, the elements of the multiplier ideal are the functions that one has to multiply sections of L with to ensure expected positivity properties, hence the adjective "multiplier".)

While at first sight these generalizations may seem to be technical (even downright pointless) improvements, they lead to an immensely powerful method with applications to many questions, due mainly to Ein, Lazarsfeld, Siu and their coworkers. After an outline of the basic theory in Chapter 9, several of these applications to zeros of theta functions, to Hilbert's Nullstellensatz and to effective versions of very ampleness criteria are treated in Chapter 10.

Chapter 11 is devoted to an asymptotic version of multiplier ideals which seems to be an especially efficient way to focus on the key positivity properties of line bundles. An application shows that the dual of the cone of effective divisors $NE_{n-1}(X^n) \subset H_{2n-2}(X^n, \mathbb{R})$ is spanned by those curves whose deformations cover a dense subset of X, returning us to the study of duality between curves and line bundles begun in Chapter 1. This gives a feeling of completeness and harmony, until we realize that while we learned much about curves (= dimension 1) and line bundles/divisors (= codimension 1) on a variety, very little was revealed about the intermediate dimensions.

This is, however, a problem that pervades all algebraic geometry and should be one of the main challenges for the next generation of researchers.

Finally, a comment about the marketing and price of these books, which should be of interest to all future authors.

From the business point of view an unusual, and probably closely watched, publishing decision was that the softcover edition appear simultaneously with the hardcover one. While I was not party to the discussions, I know that Lazarsfeld achieved this only after lengthy negotiations and, I suspect, not without making sacrifices. If this model of publishing becomes a success, it would help considerably in moderating the high cost of scientific publications.

References

[Deb01] Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR1841091 (2002g:14001)

[GY86] L. Zhiyong Gao and S.-T. Yau, *The existence of negatively Ricci curved metrics on three-manifolds*, Invent. Math. **85** (1986), no. 3, 637–652. MR848687 (87j:53061)

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. MR1658959 (2000b:14018)

[Laz04a] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. MR2095471 (2005k:14001a)

[Laz04b] ———, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. MR2095472 (2005k:14001b)

[Mat02] Kenji Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002. MR1875410 (2002m:14011)

[Sul04] Dennis Sullivan, *René Thom's work on geometric homology and bordism*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 3, 341–350 (electronic). MR2058291

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