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Embedding problems in symplectic geometry, by Felix Schlenk, de Gruyter Expositions in Mathematics, vol. 40, Berlin, 2005, x+250 pp., US\$99.95, ISBN 3-11-017876-1

There is currently a great deal of research on symplectic manifolds. To someone outside the field, natural questions are: What is a symplectic manifold? What makes them interesting? To get a flavor of this geometry, it is first important to point out that symplectic geometry is very different from Riemannian geometry. The local structure of a symplectic manifold is always equivalent to the standard structure on Euclidean space, and hence there are no local invariants, such as curvature, for a symplectic manifold. Thus in a spirit similar to topology, one is led to the study of global phenomena. One example of such a global question is to understand what can or cannot be done with symplectic embeddings. For example, what can be said about the shape of the symplectic image of a ball? It turns out that at times symplectic embeddings are quite rigid, while at other times very flexible. The transition between rigidity and flexibility is fascinating and not well understood. Rigidity results often stem from the theory of pseudo-holomorphic curves introduced by Gromov in 1985. Flexibility results often arise from explicit constructions. This book gives a nice overview of rigidity phenomena but focuses on the art of constructing symplectic embeddings through the techniques of "symplectic folding", "multiple symplectic folding", "symplectic wrapping", and "symplectic lifting".

An elementary symplectic manifold is $\mathbb{R}^{2n} = \{(x_1, y_1, \dots, x_n, y_n)\}$ equipped with the closed, nondegenerate 2-form $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Any open subset of $\mathbb{R}^{2n}$ is a symplectic manifold. Given two open subsets U, V of $\mathbb{R}^{2n}$, an embedding $\varphi : U \to V$ is called symplectic if $\varphi^*\omega_0 = \omega_0$. A basic problem in symplectic geometry is to fix two open sets U, V and understand when there exists a symplectic embedding of U into V.

An elementary property of symplectic embeddings is that they must preserve volume. So, for example, there does not exist a symplectic embedding of a 4-dimensional ball of volume 1 into a 4-dimensional ellipsoid of volume A when $A < 1$. However, volume-preserving embeddings are quite flexible: Given open and connected subsets U, V of $\mathbb{R}^{2n}$, there exists a volume-preserving embedding of U into V if and only if the volume of U is less than or equal to the volume V. In dimension 2, symplectic embeddings coincide precisely with volume-preserving embeddings, and in higher dimensions for many years there were believers that whatever could be done by a volume-preserving diffeomorphism could be approximately also done by a symplectic diffeomorphism. In the early 1970's, Gromov proved the following "hard versus soft" alternative:

Theorem 1 (Gromov's Alternative)**.** *The group of symplectomorphisms of $\mathbb{R}^{2n}$ is either C^0-closed in the group of all diffeomorphisms (hardness or rigidity), or its C^0-closure is the group of volume-preserving diffeomorphisms (softness or flexibility).*

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We now know that hardness holds. One of the most geometric expressions of symplectic rigidity is Gromov's Nonsqueezing Theorem. To state this result, let $B^{2n}(r) \subset \mathbb{R}^{2n}$ be the open ball of radius $\sqrt{r}$:

$$B^{2n}(r) = \{(x_1, y_1, \ldots, x_n, y_n) : x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2 < r\},$$

and let $C^{2n}(R) \subset \mathbb{R}^{2n}$ be the "symplectic cylinder" of radius $\sqrt{R}$:

$$C^{2n}(R) = \{(x_1, y_1, \ldots, x_n, y_n) : x_1^2 + y_1^2 < R\} = B^2(R) \times \mathbb{R}^{2n-2}.$$

Since the cylinder $C^{2n}(R)$ has infinite volume, for any r there exists a volume-preserving embedding of $B^{2n}(r)$ into $C^{2n}(R)$. However,

Theorem 2 (Gromov's Nonsqueezing)**.** *There exists a symplectic embedding of the ball $B^{2n}(r)$ into the cylinder $C^{2n}(R)$ if and only if $r \leq R$.*

In particular, this says that if one can symplectically embed a ball into such a cylinder, then one can use a translation as the embedding. This result makes symplectic embeddings seem quite rigid. A natural question is then if $r > R$, how much of the ball can be put into the cylinder? A surprising answer is that for any $r > R$, it is possible to construct a symplectic embedding of $B^{2n}(r)$ into $\mathbb{R}^{2n}$ so that an arbitrarily large percentage (less than 100%) of the volume of the $B^{2n}(r)$ will have its image in the cylinder. Gromov deduced this Nonsqueezing Theorem from the theory he developed of pseudo-holomorphic curves, [G]. The contrasting flexibility statement about the existence of embeddings can be proved using results in this book.

Some other interesting embedding problems come from looking at a target space with finite volume. For example, one can fix an ellipsoid and then ask what is the smallest ball into which this ellipsoid will symplectically embed. The ellipsoid $E(a_1, \ldots, a_n) \subset \mathbb{R}^{2n}$ is defined as

$$E(a_1, a_2, \ldots, a_n) = \left\{(x_1, y_1, \ldots, x_n, y_n) : \frac{x_1^2 + y_1^2}{a_1} + \cdots + \frac{x_1^2 + y_1^2}{a_n} < 1\right\}.$$

In [FHW], as an application of symplectic homology, Floer, Hofer, and Wysocki found that a "fat" 4-dimensional ellipsoid cannot embed into certain balls even though the volume restriction is satisfied. More precisely,

Theorem 3. *Assume $a_2 \geq a_1 \geq \frac{a_2}{2}$. Then the ellipsoid $E(a_1, a_2)$ does not symplectically embed into the ball $B^{2n}(A)$ if $A < a_2$.*

In particular, this says that if such a "fat" ellipsoid can be embedded into a ball, then one can use a translation to put it there. Is there more flexibility with "skinny" ellipsoids? In fact, yes. Using a "wrapping construction" as developed in [T], one can show, in particular, that there does exist a symplectic embedding of $E(1, 12)$ into $B^4(4)$. By a "folding construction" developed in [LM], Lalonde and McDuff show that, in particular, there exists a symplectic embedding of $E(1, 4)$ into $B^4(3 + \epsilon)$, for all $\epsilon > 0$. Depending on the ellipsoid, sometimes the wrapping construction will place the ellipsoid into a smaller ball, while at other times the folding construction will be more space efficient.

In this book, Schlenk uses the theory of capacities, [EH], to extend the above nonembeddability result of fat ellipsoids to higher dimensions, and he extends the above embeddability result of thin ellipsoids to higher dimensions by a refinement of Lalonde and McDuff's folding construction.

Theorem 4. *If $a_n \geq a_1 \geq \frac{a_n}{2}$, then the ellipsoid $E(a_1, \ldots, a_1, a_n)$ does not symplectically embed into $B^{2n}(A)$ when $A < a_n$.*

Theorem 5. *Assume $\frac{a_n}{2} > a_1$. Then there exists a symplectic embedding of the ellipsoid $E(a_1, \ldots, a_1, a_n)$ into the ball $B^{2n}(a_n - \delta)$ for every $\delta \in (0, \frac{a_n}{2} - a_1)$.*

Note that these higher-dimensional ellipsoids are "flat" in the smaller radii. Ellipsoids which are flat in the larger radii can be more rigid: the third Ekeland-Hofer capacity implies that for $n \geq 3$, the $2n$-dimensional ellipsoid $E(a, 3a, \ldots, 3a)$ does not symplectically embed into the ball $B^{2n}(A)$ if $A < 3a$. A capacity argument also shows that the "nonflat" ellipsoid $E(a, 2a, 3a)$ does not embed into the ball $B^6(A)$ if $A < 2a$, but the following is unknown:

Question 6. Does the ellipsoid $E(a, 2a, 3a)$ symplectically embed into $B^6(A)$ for some $A < 3a$?

Another variation of embedding problems is to consider symplectic embeddings of a collection of objects. Given a $2n$-dimensional symplectic manifold (M, ω) of finite volume, the symplectic packing problem asks if given a natural number k, what is the largest number a for which the disjoint union of k equal balls $B^{2n}(a)$ symplectically embeds into (M, ω)? Via his theory of pseudo-holomorphic curves, Gromov found obstructions to symplectic packings of a ball. In particular, when trying to embed two 4-dimensional balls into a larger 4-dimensional ball, one can cover at most half the volume of the larger ball, [G]. In [MP], McDuff and Polterovich extended this result and were able to determine the maximal packing densities when packing nine or less smaller balls into a larger ball. The calculations of the maximal densities, p_k, $k = 1, \ldots, 9$, are given by the following chart:

k	1	2	3	4	5	6	7	8	9
p_k	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{4}{5}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

McDuff and Polterovich's proof of the existence of these packings is nonexplicit; they result from the symplectic blow-up operation. It is possible to see some of these maximal packings via a wrapping or folding construction. For example, Schlenk explains in his book how the following figure, Figure 1, represents a maximal packing of the 4-dimensional ball by 6 balls; any 5 of these shaded objects represents a maximal packing via 5 balls. It is still an open problem to find an explicit construction for the maximal packings via 7 or 8 balls. As a surprising contrast to the above packing obstructions, Biran showed that there are no packing obstructions when packing more than nine balls into a larger ball, [B]. In other words, the maximal packing density p_k of the 4-dimensional ball equals 1 for all $k > 9$.

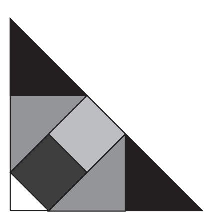

FIGURE 1. A maximal packing of the 4-dimensional ball by 6 balls.

Overall, this book is a welcome addition to the mathematics literature. The exposition is self-contained, and the only prerequisites are a basic knowledge of differential forms and smooth manifolds. The book is addressed to mathematicians interested in geometry or dynamics but may also be useful to physicists working in a field related to symplectic geometry. Deep motivations for and results related to particular embedding problems are given throughout the book, and so the reader learns a great deal about the field of symplectic topology. The particular constructions are an interesting combination of analysis and geometric combinatorics. Clear explanations make the constructions easy to read; analytic details are thorough. There are some very interesting computer calculations for optimal embeddings of ellipsoids into balls and cubes via multiple foldings. Also some intriguing comparisons are made between the folding and wrapping techniques. Through the many embeddings problems considered and the many constructions explained in the book, the reader gets a very hands-on introduction to what can be done with symplectic embeddings. The particular constructions described may have applications well beyond the particular results considered in the book.

Indeed embedding questions are still active areas of research. One new area where embeddings are being considered is in contact geometry, the sibling of symplectic geometry. Recently Eliashberg, Kim, and Polterovich have discovered some parallels and nonparallels to Gromov's Nonsqueezing Theorem in the contact setting, [EKP]. To explain this, first consider the elementary contact manifold $(\mathbb{R}^{2n} \times S^1, \xi)$ where $S^1 = \{t \in \mathbb{R}/\mathbb{Z}\}$ and ξ is the maximally nonintegrable field of hyperplanes given by $\ker(dt - \alpha)$ where α is the Liouville form $\sum \frac{1}{2}(x_i dy_i - y_i dx_i)$. Rather than looking at embeddings, in this setting it is more interesting to consider "squeezings": For an open set U, let $\overline{U}$ denote its closure, and then given open subsets $U, V \subset \mathbb{R}^{2n} \times S^1$, say U can be *contactly squeezed* into V if there exists an isotopy $\psi_t : \overline{U} \to \mathbb{R}^{2n} \times S^1$, $t \in [0,1]$, such that $(\psi_t)_* \xi = \xi$ for all t, $\psi_0 = id$, and $\psi_1(\overline{U}) \subset V$. In parallel to the ball and cylinder $B^{2n}(R_1), C^{2n}(R_2) \subset \mathbb{R}^{2n}$ described above, we now consider

$$\widehat{B}^{2n}(R_2) = B^{2n}(R_2) \times S^1, \qquad \widehat{C}^{2n}(R_1) = C^{2n}(R_1) \times S^1.$$

Then there is the following contact analogue to Gromov's Nonsqueezing Theorem:

Theorem 7 (Contact Nonsqueezing). *If there exists $m \in \mathbb{N}$ so that $R_1 \leq m \leq R_2$, then $\widehat{B}^{2n}(R_2)$ cannot be squeezed into $\widehat{C}^{2n}(R_1)$.*

However, as a contrasting flexibility result, if $2n \geq 4$ and $R < 1$, then $\widehat{B}^{2n}(R)$ *can* be squeezed into $\widehat{C}^{2n}(\epsilon)$ for any $\epsilon > 0$. Again, understanding the transition from rigidity to flexibility is challenging. The following question from [EKP] indicates that, as in symplectic geometry, embedding questions are alive and well in contact geometry:

Question 8. Given $m \in \mathbb{N}$ and R_1, R_2 such that $m < R_1 < R_2 < m+1$, can $\widehat{B}^{2n}(R_2)$ be squeezed into $\widehat{C}^{2n}(R_1)$?

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