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The mathematics of the Bose gas and its condensation, by Elliott H. Lieb, Robert Seiringer, Jan Philip Solovej, and Jakob Yngvason, Birkhäuser-Verlag, Basel, 2005, viii+208 pp., US$39.95, ISBN 978-3-7643-7336-8

Non-relativistic quantum mechanics is now about eighty years old and is one of the most successful theories in physics. But it is conceptually difficult, and in early times it was already clear that progress in quantum mechanics depends crucially upon mathematics and also upon progress in very different fields of mathematics. It also happened several times and will continue to happen that problems in quantum mechanics initiate some mathematics which then become a field of great activity in pure or applied mathematics. Take for instance the spectral theory of Schrödinger operators or the large number of activities in pure mathematics related to "Quantum Chaos".

Understanding the properties of matter is one of the central problems in quantum mechanics, and therefore one usually has to investigate the quantum-mechanical many-body problem.

The book under review, LSSY, written by leading experts in the field, deals with a very interesting problem of many-body quantum mechanics, namely the Bose gas and one striking related phenomenon, the Bose-Einstein condensation.

First a few remarks about bosons and fermions seem appropriate. Consider a many-body non-relativistic Schrödinger operator describing a system of N interacting d-dimensional particles.

$$H_N = -\sum_{i=1}^{N} \mu_i \Delta_i + \sum_{1 \le i < j \le N} v(x_i, x_j) \tag{1}$$

defined on $L^2(\mathbb{R}^{dN})$. Here x_i denotes the position of the i-th particle and Δ_i is the associated Laplacian. The μ_i are usually related to the mass m_i of the i-th particle, ($\mu_i = \hbar^2/2m_i$). The potential $v(x_i, x_j) : \mathbb{R}^{2d} \mapsto \mathbb{R}$ describes the interaction between pairs of particles. If one considers Coulombic systems, then $v(x_i, x_j) = \alpha_i \alpha_j |x_i - x_j|^{-1}$ where α_i is the charge of the i-th particle; the system is then neutral if $\sum \alpha_i = 0$. If v is sufficiently well behaved, then H_N is well defined and bounded from below if viewed as a quadratic form. The form domain is implied by the physics of the problem. The infimum of the spectrum is denoted by $E_0(N)$ and depends of course on the chosen domain.

Roughly speaking, bosons and fermions can be described as follows: Assume for simplicity that we have two kinds of particles and that a wave function $\psi(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N)$ describes the k bosons with positions $x_1, \ldots, x_k$ and the remaining $N - k$ fermions with positions $x_{k+1}, \ldots, x_N$. Then ψ is symmetric with respect to permutation of any of the first k coordinates and antisymmetric, i.e., changes sign, if we interchange any two coordinates of the remaining particles. There is no rule concerning interchange of coordinates of the first k particles and the remaining ones.

If v is the Coulomb potential, then $E_0(N) \sim -N$ for $N \to \infty$ for usual matter (nuclei and fermionic electrons). This has been shown in the celebrated papers by

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Dyson and Lenard [3] and Lieb and Thirring [5]. If the electrons would also be bosons, $E_0(N) \sim -N^{7/5}$, a deep result of Dyson [4] (upper bound) and Conlon, Lieb, Yau [2].

Real matter consists of positively charged nuclei and negatively charged fermionic electrons. The $N^{7/5}$ behaviour fortunately does not occur in everyday life; otherwise there would be a big collapse.

But there are particles or composite particles which are bosons. In particular some atomic gases can be modeled by a Hamiltonian like (1), but with $v(x)$ not Coulombic. The particles are neutral atoms, and the corresponding system of N interacting d-dimensional particles is modeled by the N-particle Hamiltonian

$$H_N = -\mu \sum_{i=1}^{N} \Delta_i + \sum_{1 \le i < j \le N} v(|x_i - x_j|) \tag{2}$$

on $L^2(\Lambda^N)$. Here Λ is a d-dimensional cube with side length L in which each particle lives. The boundary conditions are then periodic or Neumann conditions in Λ^N. The potential is some non-negative function $v(|x|)$ which decays sufficiently fast and $\mu = \hbar^2/2m$ where m is the mass of the atom.

In experiments the atoms are usually in a trap, and this is modeled by the Hamiltonian

$$H_N^{\text{trap}} = \sum_{i=1}^{N} (-\Delta_i + V(|x_i|)) + \sum_{1 \le i < j \le N} v(|x_i - x_j|) \tag{3}$$

on $L^2(\mathbb{R}^{dN})$. The confining potential $V(|x|)$ tends to $+\infty$ as $|x|$ tends to ∞. For certain problems one can replace the $-\Delta_j$ by $(i\nabla_j + A(x_j))^2$ with A a vector potential in order to describe rotating Bose gases and superfluidity.

Estimates related to these operators are investigated in detail in LSSY. This book comes in time, since in the last few years there has been tremendous activity in theoretical and mathematical physics related to these operators. This is not surprising, since Bose-Einstein condensation was verified by impressive experiments in the mid-nineties.

The starting point of LSSY are estimates of the bosonic groundstate energy $E_0(N)$ of the Hamiltonian (2). Thereby the thermodynamic limit (recall that L is the side length of the cube Λ),

$$e_0(\rho) = \lim_{L \to \infty} \frac{E_0(N, L)}{N}, \ \ \rho = N/L^d \text{ fixed},$$

is the physically interesting one. For $d = 3$ and $d = 2$ upper and lower bounds to $e_0(\rho)$ are obtained, and the differences between the three- and the two-dimensional case are explained in detail. Here the scattering length a associated to the potential v plays a crucial role.

One of the major goals of the book is to understand the Bose-Einstein condensation, BEC. This phenomenon was predicted by Einstein even before quantum mechanics in its present form existed. He extended Bose's observations concerning the statistics of photons to massive particles, i.e., cold dilute gases which he modeled by non-interacting bosons. Then, roughly speaking, BEC means that the one particle groundstate is macroscopically occupied. For interacting systems this does not generalize directly, but there is an intriguing definition of BEC. Assume that a bosonic system described by a Hamiltonian $H(N)$ such as given in (2) or (3) has a normalized groundstate ψ_0 with groundstate energy $E_0(N)$ so that

$$H(N)\psi_0(x_1, x_2, \ldots, x_N) = E_0(N)\psi_0(x_1, x_2, \ldots, x_N).$$

Without loss we might assume ψ to be real valued. One defines BEC or says in other words that there is condensation if the one particle density matrix $\gamma_N(x, x')$,[1]

$$\gamma_N(x, x') = N \int_{\mathbb{R}^{3(N-1)}} \psi_0(x, x_2, \ldots, x_N)\psi_0(x', x_2, \ldots, x_n) dx_2 \ldots dx_N,$$

viewed as an integral operator, has the property that its largest eigenvalue satisfies $\lambda(N) \geq cN$ for some $c > 0$ as N tends to infinity. Of course

$$tr(\gamma_N(x, x)) = \int_{\mathbb{R}^{3N}} |\psi_0|^2 dx_1 \ldots dx_N = N.$$

The proof of BEC according to this definition has not been achieved yet for the thermodynamic limit. But for an important special case BEC was verified as explained in LSSY.

It can be shown under specific assumptions for v that the Schrödinger equation associated to H_N^{trap} tends as $N \to \infty$ in a suitable sense to a non-linear effective one-particle equation, the Gross-Pitaevskii equation. The corresponding functional reads

$$\mathcal{E}\{\Phi\} = \int_{\mathbb{R}^3} \Big(|\nabla\Phi|^2 + V|\Phi|^2 + 4\pi a|\Phi|^4\Big) dx, \;\; \int_{\mathbb{R}^3} |\Phi|^2 dx = N,$$

where a is again the scattering length associated to v. This functional is closely related to BEC and also to its definition via the largest eigenvalue of the density matrix γ_N.

The last few paragraphs clearly show that LSSY concentrates on some questions which are of great relevance in theoretical physics. Most of the results are rather recent. In fact the book grew out of a course given in Oberwolfach in 2004 and is an extended pedagogical version of 7 papers written by the authors since 1998. Just a few words about the other topics considered in LSSY. There is a very interesting chapter on bosonic one- and two-component dense Coulomb systems in which some old conjectures from the sixties are resolved. There is also a chapter on condensation for a lattice Bose gas which is described by a discrete model.

The authors take great pains to motivate and explain the different scalings and the underlying physical principles. This is very helpful and absolutely necessary, since they explain a field which is growing fast and where some conceptual questions are still being discussed. It is also not easy to keep track of all the different scaling limits: the thermodynamic limit, the low and the high density limits, the Thomas-Fermi limit (which has nothing to do with the Thomas-Fermi theory for atoms), and the Gross-Pitaevsky limit and some of these limits for different dimensions, i.e., $d = 3, 2$ and sometimes $d = 1$. But these complications are inevitable and make LSSY so interesting, since the modeling and the mathematical physics happen partly simultaneously.

There are also other activities related to this field, demonstrating that the theoretical questions related to BEC already have some mathematical spin-off. For instance a version of the Gross-Pitaevskii equation describes rotating Bose gases where vortices occur, perhaps similar to the superconducting systems described by

[1] The factor N in the definitions is the usual convention.

the Ginzburg-Landau equation. A recent book is entirely devoted to this subject [1].

On the other hand there are the ideas and insights related to Bogoliubov's work dating back to 1947, which play a big motivating role in the field, but there are only a few cases where these insights have been rigorously verified. Those are discussed in LSSY; in addition some non-rigorous ideas are used to give many problems an attractive basis in physics. See Appendix A and some discussions at the ends of the chapters, where very interesting partly non-rigorous theoretical physics are presented. This is also reflected by the references. About two thirds of the cited papers are clearly papers in theoretical and experimental physics, and the rest are mathematical physics.

The mathematics which is presented is very concrete hard analysis. For many of the estimates some physical insight is crucial. This makes the reading not easy, but this is inevitable.

LSSY addresses primarily mathematical physicists. The book could be the basis of mathematical physics seminars. Furthermore one finds there many intriguing mathematical problems of independent interest as well as mathematical techniques which are either new or difficult to find in the literature. This will be of interest for mathematical analysts. LSSY definitely will play an important role in research related to this dynamic topic in mathematical physics.

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T. Hoffmann-Ostenhof

Institut für Theoretische Chemie, Universität Wien, and International Erwin Schrödinger Institute for Mathematical Physics

E-mail address: thoffman@esi.ac.at