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Self-similar groups, by Volodymyr Nekrashevych, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005, xii + 231 pp., US$59.00, ISBN 978-0-8218-3831-0

Self-similarity, or fractalness, is an important phenomenon in nature that is reflected in many aspects of modern civilization. It appears in physics, chemistry, biology, the medical sciences, computer science, as well as in art. Self-similarity appears in various topics of mathematics and mathematical modeling, involving dynamical systems and chaos, random processes and statistical physics, topology and fractal geometry. It deals with systems covered by patterns, or tiles, which appear indefinitely in different scales of magnitude. The use of self-similarity in mathematics is mostly based on the re-scaling or renormalization principle and is formulated in terms of the renorm group. The idea of scale invariance is old in mathematics and physics. Scaling arguments were known to the Pythagorean school, Euclid and Galileo. The renormalization (or shortly renorm) group has made its appearance in different places and refers to a set of techniques and concepts related to the change of a physical or mathematical model depending on the change of the observation scale.

The renorm group in the classical case is a cyclic group or a continuous one-parameter group (isomorphic to the additive group of reals), such as the group generated by the adding machine (or odometer) or a group of specific transformations of a partial solution of a mathematical/physical problem. Indeed, in practice it is a semigroup (cyclic or one-parametric), as very often the transformations do not have an inverse [Wik06, Shi00].

However, in more modern times, new self-similar structures have been discovered having non-commuting renorm group transformations, with complicated algebraic properties, reflecting the properties of the model itself. This transition from cyclic renormalization to non-commutative renormalization can be compared to the transition from classical geometry to non-commutative geometry [Con94].

Non-commutative geometry uses the methods and techniques of operator algebras. Recent developments show that non-cyclic renormalization also involves consideration of C^*-algebras.

The class of groups that lies behind non-cyclic renormalization is that of self-similar groups (self-similar semigroups arise naturally as well). This is a new, quickly developing area of modern group theory related to many topics in geometric group theory, asymptotic group theory, and Galois theory and is also closely related to the theory of automaton groups or groups generated by finite automata. Automata here are understood in the sense of finite input-output Turing machines or Mealy automata and not in the sense of recognition automata, as in the automatic groups [ECH92].

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There are many ways to define self-similar groups. One way is, as was just mentioned, to define them as groups generated by Mealy type automata. This makes self-similar groups suitable for needs of computer science. Another way to define them is via actions on the space of sequences over a finite alphabet, which brings them close to various topics in dynamical systems. It is well known that rooted trees and dynamics on rooted trees in the form of adding machines (or odometers) appear in various situations in dynamical systems and chaos. See, for instance, the books [BOERT96] and [Bue97], which describe this situation comprehensively. However, the variety of examples of such links and relations is much broader.

Let us give a hint as to the notion of a self-similar group. In general terms, if a group G acts on a self-similar set Y and Z is a subset similar to Y, then it is required that the stabilizer of Z in G induces on Z a group isomorphic (or geometrically similar) to G. A rigorous definition will be given later, but now let us explain the role of self-similar groups and self-similar actions in group theory. This requires a few words about actions on rooted trees, as they play a fundamental role in the theory.

A rooted d-regular tree T_d is perhaps the most self-similar object, since, for every vertex u, the rooted subtree of T_d with root at u is canonically isomorphic to T_d. Note that the rooted d-regular tree T_d is a subtree of the homogeneous $(d+1)$-tree in the well-known Bass-Serre-Tits theory. An important difference is that in Bass-Serre-Tits theory a common assumption is the smallness (in various senses) of the stabilizers of vertices (see [Ser80]), while in the case of actions on rooted trees the whole group stabilizes the root vertex.

The theory of groups acting on rooted trees is closely related to the theory of iterated wreath products initiated by L. Kaloujnin and P. Hall. At the same time, this is a geometric version of the theory of residually finite p-groups when $d = p$ is a prime number. The nicest class of actions on rooted trees is the class of self-similar actions, which often provide wonderful examples of groups. For instance, some of the basic examples in the field include: the first examples of groups of intermediate growth; Gupta-Sidki examples of finitely generated torsion p-groups; P. Neumann examples of groups with unusual structure of the lattice of subnormal subgroups; J.S. Wilson examples of groups of exponential but not uniformly exponential growth; Šunić examples of Hanoi groups $H^{(k)}$, related to the Hanoi Towers game; and Nekrashevych examples of iterated monodromy groups, including the Basilica and Sierpinski groups. Even well known groups, such as free abelian groups, non-commutative free groups, the lamplighter group, and many others, when realized as self-similar groups bring new vision to some classical results (like the theory of numeration systems and tilings) and sometimes provide a solution to known problems (as in the case of the lamplighter group and the Atiyah Problem on L_2-Betti numbers).

Each of the above examples represents a famous problem or a direction in mathematics. Gupta-Sidki p-groups and other related examples represent an elegant (and perhaps the shortest) approach to the solution of the General Burnside problem; groups of intermediate growth solved a question of J. Milnor and opened a new direction in the study of asymptotic properties of groups and manifolds; these groups as well as the Basilica group and many other examples of self-similar groups prove successful in the study of amenable groups, answering a question of M. Day and related problems; Hanoi Groups shed new light on the classical Hanoi Towers Problem involving more than three pegs; and this list goes on and on.

Self-similar groups find applications in Galois Theory (for instance when one considers towers of field extensions attached to the roots of polynomials that are iterations of a single polynomial). Another set of examples of applications comes from holomorphic dynamics and the theory of covering manifolds, where self-similar groups and some associated geometric objects such as Schreier graphs and limit spaces are related to Julia sets.

It was clearly time to have a book that describes these and other topics involving self-similar groups. The only previous book that touched a bit on this field is that of P. de la Harpe [dlH00], but only one chapter is devoted to the present topic. In reality, the book of Volodymyr Nekrashevych is the first monograph on the subject of self-similar groups and their applications. It is mostly based on original results of the author, who has made fundamental contributions to the area.

The book starts with a preliminary chapter that contains the basic definitions and examples and also gives a quick overview of the subject. Here the basic notions and tools of the theory appear: rooted trees and actions on rooted trees, finite automata and Moore diagrams, wreath actions and branch groups, adding machines and bi-reversible automata, and the main definition of self-similar actions and self-similar groups. Many examples, including the cyclic groups realized by the adding machine, examples of torsion groups of intermediate growth, the groups of exponential but non-uniformly exponential growth, and sophisticated realizations of free groups and some free products by means of bi-reversible automata, are also presented.

An object that perhaps best of all reflects self-similarity (as was already mentioned) is the d-regular rooted tree $T = T_d$, for $d \geq 2$. For any vertex u the subtree T_u rooted at u is isomorphic to the whole tree. The tree T_2 is a tree-like model of the classical middle third Cantor set C. Let $X = \{x_1, \ldots, x_d\}$ be an alphabet consisting of d letters, and let X^* be the set of words over X (which can also be viewed as a free monoid on d generators). The vertices of T_d correspond bijectively to the elements of X^*, and the boundary ∂T_d of the tree consisting of geodesic rays joining the root vertex with infinity is then in a natural bijection with the space X^ω of infinite sequences of symbols from the alphabet X. We see that any of the sets T, X^*, C or X^ω is a natural candidate for a set on which self-similar actions can be defined. There is no difference no matter which one is used in the definition that follows.

Definition. A faithful action of a group G on X^ω is said to be *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w),$$

for all $w \in X^\omega$.

A group is called self-similar if it is represented by a self-similar action. Such a representation is not unique and is not always possible (among the restrictions is the solvability of the word problem and residual finiteness).

A crucial feature of self-similar automaton groups is that they are groups generated by the states of one (non-initial) automaton. The case when the automaton is finite is of special interest, as the most interesting self-similar groups are usually groups generated by finite automata. A classification of such groups can be based on a classification of finite automata. For instance, following S. Sidki, one can define the notion of growth of a finite automaton (which can be bounded, polynomial or exponential) and study the corresponding classes of groups. There is an alternative approach to the growth of finite automata having roots in Milnor's notion of growth in finitely generated groups, which leads to examples of automata (and groups) of intermediate growth that were already mentioned.

One of the main problems related to self-similar groups is to determine which groups can have faithful self-similar actions and to find suitable self-similar realizations.

There are other important notions that play a crucial role in the theory of self-similar groups. Among them let us list spherical transitivity (or level transitivity) of the action, recurrence (not in the probabilistic sense; self-replication would perhaps be a better term), boundedness, branching, and contraction. We refer the reader to the book for these notions, but give some hints regarding them.

Level transitivity means transitivity of the action on all levels of the tree. It is equivalent to ergodicity of the induced action on the boundary ∂T of the tree supplied with the uniform Bernoulli measure. Being recurrent ("self-replicating") means that for any vertex u of the tree the restriction of the action of the stabilizer of u in G onto the subtree T_u is equal to the whole group, as long as we identify the tree T_u with the whole tree T by the natural identification (in general, the stabilizer of u in the case of a self-similar action is just a subgroup of G). Boundedness is a condition on the number of non-trivial actions on the subtrees growing from the vertices on a given level. Branch (or more generally weakly branch) groups are groups acting faithfully and level-transitively on a spherically homogeneous rooted tree in such a way that the structure of the lattice of subnormal subgroups mimics the structure of the tree. This is a very rough hint to the actual definition (for more details see [BGŠ03]). Finally, a group G is contracting if it is finitely generated and its projections g_u have strictly smaller length (with respect to the chosen system of generators) for all sufficiently long $g \in G$.

We have spent some time outlining the basic notions concerning self-similar groups, and now we are ready to review some of the most important topics presented in the book.

The first chapter contains the basic definitions and examples including the multidimensional adding machines, a torsion 2-group of intermediate growth (which we denote here by $\mathcal{G}$), Gupta-Sidki p-groups, groups of P. Neumann type, and groups of J.S. Wilson type, which have exponential but not uniformly exponential growth. Also such well known examples in group theory as the wreath product of the infinite cyclic group and the group of order two (the lamplighter group), free groups and free products of groups of order 2, realized as self-similar groups, appear in Chapter 1. Besides that, important classes of automata such as bounded automata and bi-reversible automata are considered here and are used in the construction of the corresponding groups. The variety of examples gives the reader an impression of the richness of the theory of self-similar groups and provides a quick entrance into the subject.

In the second chapter the basics of the algebraic theory of self-similar groups are explored. The chapter starts with the definition of a permutational bimodule and association of such a bimodule to a self-similar action. This notion was introduced by the author and happens to be quite successful, as it allows one to describe in categorical language many of the notions and ideas involving self-similar groups.

In particular, it is useful in the study of limit spaces, representations, orbifolds, C^*-algebras and many other objects associated to self-similar groups.

The author also considers tensor products of bimodules and uses them to study the tensor powers of self-similar actions. Tensor products play an essential role in many places in the book.

The next important objects of study in the book are the virtual endomorphisms of self-similar groups and related objects (such as associated covering bimodules). Virtual endomorphisms already appeared in M. Shub's thesis of 1969. This notion is related to the abstract commensurator of a group (which is a group of virtual automorphisms). For many self-similar groups it is an extremely interesting object. For instance, as was observed by C. Roever in the case of the torsion 2-group of intermediate growth $\mathcal{G}$, it is a finitely presented simple group S, generated by $\mathcal{G}$ and another subgroup of S, which is isomorphic to the very famous R. Thompson group F defined earlier in logic in the study of associativity logic.

Chapter 2 contains a complete description of level-transitive self-similar actions of free abelian groups of finite rank, which is based on joint results of the author with S. Sidki. It is shown that all such actions can be interpreted as "A-adic" groups, i.e., as generalizations of the adding machine.

An important notion that appears in Section 2.10 is the rigidity of self-similar actions and its variations based on a joint work with Y. Lavrenyuk. The rigidity is applied to show uniqueness of the action of groups of branch type (under certain conditions).

The chapter ends with two sections devoted to the study of contracting self-similar actions and related notions. Contracting actions and contracting self-similar groups comprise the most studied subclass of self-similar groups and have many applications within group theory and in other areas of mathematics. The rate of contraction is characterized by the contraction coefficient (which for contracting groups is less than 1). Self-similar finite-state actions of abelian groups are contracting. The first implication of the contraction property is the solvability of the word problem by a very efficient algorithm of branch type. Another consequence is polynomial growth degree of the orbits of the action of the group on the boundary of the tree or, equivalently, polynomial growth of the corresponding Schreier graphs. Let us stop for a moment on the notion of Schreier graphs, as they play a crucial role in many considerations.

The Schreier graph of an action on a set Ω of a group G generated by a finite symmetric system of generators S is the graph Γ with the set of vertices $V(\Gamma) = \Omega$ and the set of edges $E(\Gamma) = \{(x, sx) \ : \ x \in \Omega, s \in S\}$. In the case of a transitive action this graph is naturally isomorphic to the graph $\widetilde{\Gamma} = \Gamma(G, H, S)$, where H is the stabilizer of a point of Ω, $V(\widetilde{\Gamma}) = \{gH, g \in G\}$ is the set of cosets and $E(\widetilde{\Gamma}) = \{(gH, sgH) \ : \ g \in G, s \in S\}$. Schreier graphs are generalizations of Cayley graphs (which correspond to a free transitive action on Ω, i.e., to H being trivial). Schreier graphs did not play a large role in mathematics until recently. The development of the theory of self-similar groups showed the extreme importance of this notion in various topics, and the book of V. Nekrashevych perfectly demonstrates this.

We will return to Schreier graphs later in this review, but let us first move on to Chapter 3 and Chapter 5, which can be considered as the main chapters of the book. In Chapter 3 the author shows how a topological dynamical system called the limit dynamical system can be associated to every contracting self-similar action.

This construction is a bridge between self-similar groups and self-similar topological spaces. The converse construction, called the iterated monodromy group, is described and studied in Chapter 5.

Having a group G acting self-similarly on a rooted tree, one can draw the Schreier graphs Γ_n for $n = 1, 2, \dots$ of the action of G at the n-th level. What often happens is that the sequences of these graphs converge in a natural sense to some limit object which usually looks like a fractal set. In such a way one can get examples of the Sierpiński gasket, the Apolonnian gasket and many other complicated pictures. This is a heuristic hint to Nekrashevych's notion of the limit space $\mathcal{J}_G$ of a contracting self-similar action. The rigorous definition consists of factorization of the space $X^{-\omega}$ of left infinite sequences of letters over the alphabet X by the equivalence relation $\sim$ that can be described in terms of the Moore diagram of the corresponding automaton. The limit space $\mathcal{J}_G$ is metrizable and has finite topological dimension. It has an orbispace structure that comes from a presentation of $\mathcal{J}_G$ as a space of orbits of a proper action of G on the limit G-space $\mathcal{X}_G$, which is defined as well. The equivalence relation used in the definition of the limit space is invariant under the shift $\dots x_2 x_1 \mapsto \dots x_3 x_2$ on the space $X^{-\omega}$. Hence the shift defines a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$, and the dynamical system $(\mathcal{J}_G, \mathsf{s})$ is called the limit dynamical system of the self-similar action. The theory of the limit spaces becomes even richer by providing them with the structure of an orbispace and realizing them as boundaries of a Gromov-hyperbolic "self-similarity graph". Other important limit objects, namely limit solenoids and inverse limits of self-coverings, appear later in Chapter 5.

One of the origins of limit spaces is the connection between self-similar groups and numeration systems on $\mathbb{Z}^n$ and on $\mathbb{R}^n$. This allows interpretation of some of the classical results on numeration systems and the related digital tilings of $\mathbb{R}^n$, such as the "twin dragon", in terms of self-similar groups. In the non-commutative case the tiles related to the limit space of self-similar groups usually have a complicated fractal nature. In many cases though they have nice properties, such as connectedness and representability by subdivision rules. A rich source of examples of groups, limit spaces and tiles comes from the class of bounded automata.

Chapter 4 is concerned with orbispaces and also presents the main definitions and properties of pseudogroups of local homeomorphisms and étale groupoids. Here the author explores well known material, but from the point of view that is suitable for use in the study of self-similar actions.

Chapter 5 contains the most conceptual notion of an iterated monodromy group. Having a d-fold covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ of a "good" topological space $\mathcal{M}$ by its open subset $\mathcal{M}_1$, one can take a point $t \in \mathcal{M}$ and construct a d-regular rooted tree T with a root at t and n-th level L_n consisting of the pre-images $f^{-n}(t)$ of t with incidence relation determined by the rule that each vertex $u \in f^{-n}(t)$ is connected by an edge to $f(u) \in f^{-(n-1)}(t)$. The fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally on each of the levels L_n by the classical monodromy action. All these actions viewed together determine an action of $\pi_1(\mathcal{M}, t)$ on T by automorphisms, and the iterated monodromy group $\mathrm{IMG}(f)$ is the quotient of $\pi_1(\mathcal{M}, t)$ by the kernel of this action. It is surprising that this natural definition appeared just a few years ago in the papers of the author of the book and not in the articles at the end of the 19th century (for instance in the works of Riemann or Poincaré). It is also remarkable that the iterated monodromy group has a natural self-similarity structure. Thus, the whole machinery developed in the book for the study of self-similar actions on rooted trees can be used for the study of dynamical systems related to covering maps. First of all, this has a very strong connection to holomorphic dynamics and its main objects of study, such as Julia sets. Here the solenoids, leaves, and tiles start to play their role, and the Julia set is homeomorphic to the limit space of the corresponding iterated monodromy group.

Already the simplest maps considered in holomorphic dynamics such as the maps of the complex plane given by the quadratic polynomials $z^2 - 1$, $z^2 + i$, etc., show that the corresponding iterated monodromy groups can be extremely complicated but also amenable to study by using methods already known to researchers dealing with group actions on rooted trees.

Iterated monodromy groups may be torsion free, may have exponential, polynomial or intermediate growth, may be "classical" groups, such as $\mathbb{Z}$, or may be of branch type. It is an interesting phenomenon that many of them are amenable in the von Neumann sense (i.e., have an invariant mean), but are not elementary amenable in M. Day's sense. Thus iterated monodromy groups provide a large number of extremely interesting groups (with associated objects and notions sometimes having sophisticated names, such as the Basilica group, Hanoi Towers group, Bellaterra automaton, Münchhausen trick, etc.).

The power of the iterated monodromy group machinery is confirmed by the fact that some difficult questions (such as Hubbard's "twisted rabbit problem", solved by L. Bartholdi and V. Nekrashevych) were solved by their use. The last chapter of the book contains a summary of examples and applications of iterated monodromy groups to dynamics of post-critically finite polynomials and rational functions. It contains interesting discussions and links to combinatorial equivalences of self-coverings of the Riemann sphere and the corresponding Thurston's theorem. Kneading invariants of dynamical systems are generalized by V. Nekrashevych to kneading automata. Belyi polynomials naturally appear among the examples.

Self-similar groups appeared for the first time as a class about 25 years ago and were studied in the works of this reviewer, as well as by N. Gupta, S. Sidki, P. Neumann, A. Machi, V. Sushchansky, J.S. Wilson, P. de la Harpe, L. Bartholdi and other mathematicians. The field passed the first stage of its development remarkably both in its ability to answer various questions in group theory and by its wider applications. In addition to applications that were already mentioned, let us mention that the E. Zelmanov problem on groups of finite width and the J. Rosenblatt problem on superamenable groups were also solved by using self-similar groups. The interest in this class of groups is growing not only among the specialists in group theory but also specialists in dynamical systems, operator algebras, random walks, geometry (especially fractal geometry), computer science, Galois theory, etc. The recent conferences in Oberwolfach (on Galois theory), in Palo Alto (on self-similar groups) and in Berkeley (on P vs. NP), and the initiation of a new journal *Groups, Geometry and Dynamics* (with the European Mathematical Society as the publisher) in which the area of self-similar groups will be among those dominating, confirm the growing interest in self-similar groups.

The book is well written, excellently structured, and can be easily read by beginners. I recommend this wonderful book to readers without hesitation.

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Rostislav Grigorchuk
Texas A&M University
E-mail address: grigorch@math.tamu.edu