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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
DAVID EISENBUD

MR0086071 (19,119a) 18.0X

Serre, Jean-Pierre

Sur la dimension homologique des anneaux et des modules noethériens. (French)

Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, *pp.* 175–189. *Science Council of Japan, Tokyo*, 1956.

The author gives an exposition of the results of M. Auslander and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38; MR0075190 (17,705b)] and completes these results, notably by giving a homological characterization of regular local rings.

All the rings A considered in the paper are commutative noetherian rings with identity element, and all A-modules E are assumed finitely generated and unitary. If A is a local ring with unique maximal ideal $\mathfrak{m}$, and E is an A-module, a sequence of elements $a_1, \cdots, a_q$ in $\mathfrak{m}$ is called an E-sequence if for each $i = 1, \cdots, q$, a_i is not a zero divisor for $E/(a_1, \cdots, a_{i-1})E$. It is shown that every E-sequence can be extended to a maximal E-sequence, and if $a_1, \cdots, a_q$ is an E-sequence, then

$$\mathrm{dh}_A(E/(a_1, \cdots, a_q)E) = \mathrm{dh}_A E + q,$$

where $\mathrm{dh}_A(E)$ is the projective (or homological) dimension of E over A [H. Cartan and S. Eilenberg, Homological algebra, Princeton, 1956; MR0077480 (17,1040e); here $\mathrm{dh}_A E$ is written $\dim_A E$]. Moreover, if the global dimension of A is $s < \infty$, then every maximal E-sequence has length equal to $s - \mathrm{dh}_A E$; and if A is a regular local ring of dimension n, then the global dimension of A is equal to n.

If A is an arbitrary local ring, and E an A-module, the author shows that any two maximal E-sequences have the same length. This length is called the codimension of E and denoted by $\mathrm{codh}_A E$. The utility of this concept is illustrated by applications to coherent algebraic sheaves, fractional divisorial ideals, and unique factorization.

The main original result of this paper is that a local ring of finite global dimension is regular. The important tool here is the construction of a free resolution of the residue field k of A which contains in it the exterior algebra complex (loc. cit., Chapter VIII) generated over A by a minimal generating set of $\mathfrak{m}$. Using this resolution, the author shows that the (linear) dimension of ${\mathrm{Tor}_p}^A(k,k)$ is greater than or equal to $\binom{n}{p}$ where $n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ (linear dimension) [for a better estimate of this dimension see the paper reviewed below [MR0086072 (19,119b)]]. As a result of this characterization of regular local rings, the author proves that the ring of quotients $A_{\mathfrak{p}}$ of a regular local ring A with respect to a prime ideal $\mathfrak{p}$ is again regular. Other results, such as the Cohen-Macaulay theorem for regular local rings, are also proved here.

From MathSciNet, April 2007

D. Buchsbaum

MR0244243 (39 #5560) 13.95

Buchsbaum, D. A. [Buchsbaum, David A.]

Lectures on regular local rings.

Category Theory, Homology Theory and their Applications, I (*Battelle Institute Conference, Seattle, Wash.,* 1968*, Vol. One*)*, pp.* 13–32. *Springer, Berlin*, 1969.

These lectures are of triple interest, even though most of the results they contain are well known. They introduce the reader to the Koszul complex, and the way it can be applied to ring theory. They contain some examples—though standard ones—that help clarify the intimate relation between the homology of the Koszul complex and some ring properties. Finally, they constitute a brief account of the theory of local rings. The proofs of the main results generally differ from the standard ones by the repeated use of the Koszul complex. The complexes $K(x_1, \cdots, x_n)$ and $E(x_1, \cdots, x_n)$, where $x_1, \cdots, x_n$ are elements in a commutative ring R and E is an R-module, are introduced in Section 1, and the author discusses the relations between the homology groups of the complex $K(x_1, \cdots, x_n)$ and some properties of the elements $x_1, \cdots, x_n$ in the ring for the cases $n = 1, 2$. In Section 2 the R/m vector space m/m^2 is investigated for a local ring (R, m), and it is proved that $[m/m^2\colon R/m] \leq$ gl. dim R. The Hilbert-Samuel polynomial and the Artin-Rees theorem are used in Section 3 to prove that for the local ring (R, m) one has the inequality dim $R \leq [m/m^2\colon R/m]$ and the equality dim $E/(x_1, \cdots, x_s)E =$ dim $E - s$, whenever $x_1, \cdots, x_s$ is an E-sequence, for the R-module E. The Hilbert-Samuel polynomial is discussed in a somewhat more general setting than usual: Starting with a full abelian subcategory $\mathcal{A}$ of the category of R-modules, for a Noetherian ring R, the author defines the categories $\mathcal{A}_s$ for $s = 0, 1, 2, \cdots$ as the category of finitely generated graded $R[X_1, \cdots, X_s]$-modules $E = \sum_{\nu \geq 0} E_\nu$ such that (i) if $s = 0$, E_ν is in $\mathcal{A}$ for all ν; and (ii) if $s > 0$, E_ν is in $\mathcal{A}$ for all ν and the graded $R[X_1, \cdots, X_{s-1}]$ modules ker $(\sum E_{\nu_1} \overset{X_s}{\to} \sum E_\nu)$ and coker$(\sum E_\nu \overset{X_s}{\to} \sum E_\nu)$ are in $\mathcal{A}_{s-1}$.

Every function f_0 from the objects of $\mathcal{A}$ into an abelian group G that factors through the Grothendieck group of $\mathcal{A}$, induces a function f_E from the positive integers into G, setting $f_E(\nu) > f_0(E_\nu)$, for every object E in $\mathcal{A}_s$. It is proved that f_E is a polynomial function of degree less than or equal to $s-1$. Relations between codimension and homological dimension are the subject of Section 4, where it is proved that if $hd_R E < \infty$, then $\operatorname{codim} R = hd_R E + \operatorname{codim} E$. Also the chain of inequalities f.gl.dim $R = \operatorname{codim} R \leq$ dim $R \leq [m/m^2\colon R/m] \leq$ gl.dim R is established. Of significant importance is the following theorem: Let R be a Noetherian ring, E an R-module, and $\mathfrak{A}$ an ideal of R generated by elements $x_1, \cdots, x_n$ such that $E/\mathfrak{A}E \neq 0$. Let $y_1, \cdots, y_s$ be a maximal E-sequence in $\mathfrak{A}$. Then $s+q = n$, where q is the dimension of the highest non-vanishing homology of the complex $E(x_1, \cdots, x_n)$. Furthermore, $H_q(E(x_1, \cdots, x_n)) \approx (y_1, \cdots, y_s)E\colon \mathfrak{A}/(y_1, \cdots, y_s)E$.

The main properties of regular local rings are sketched in Section 5.

REVISED (January, 2007)

From MathSciNet, April 2007

A. Zaks

MR0463157 (57 #3116) 14-01

Hartshorne, Robin

Algebraic geometry.

Graduate Texts in Mathematics, No. 52.
Springer-Verlag, *New York-Heidelberg*, 1977. xvi+496 *pp*. \$24.50.
ISBN 0-387-90244-9

This text is intended to introduce graduate students to the methods and results of abstract algebraic geometry as practised today. No such exposition can succeed unless it enables the reader to make the drastic transition between the basic, intuitive questions about affine and projective varieties with which the subject begins, and the elaborate general methodology of schemes and cohomology employed currently to answer (or attempt to answer) these questions. The present text, notable for generality and depth, is also notable for its author's concern, throughout, to keep the important issues about varieties clearly in the foreground.

Varieties. An opening chapter (57 pp.) introduces affine and projective varieties, as embedded point sets. Morphisms and regular functions lead to the function field; then smooth and singular points, dimension, basic facts about smooth curves, intersections in $\mathbf{P}^n$ follow. (Examples abound. In the first 6 pages, for instance, there are 11, with 6 more in the exercises.) The chapter ends with a survey entitled "What is algebraic geometry?", aimed at motivating the subsequent developments. Here we have a discussion of several unsolved (or partly solved) problems which have motivated current as well as past research. Classification questions, both discrete and continuous, receive strong emphasis, especially moduli problems, birational classification of surfaces, and classification of singularities. (This discussion makes good expository sense, because the reader has already looked at models of one-dimensional function fields, has encountered monoidal transformations, and has blown up some embedded multiple points.) Sufficient motivation is provided by arithmetic questions to justify working over arbitrary ground fields, while reducible loci and multiple components urge acceptance of the most general coordinate rings. (This leads, in the next chapter, directly to the definition of schemes.)

Granting the value of posing the key motivating questions early, it is still obvious that no introductory text, alone, can do full justice to the work done, and being done, toward their solution. Nonetheless, there should be clear connecting links between the abstract methods to be presented and the specific questions they are intended to answer. Given the logical need to lay down general foundations first, the burden here falls largely (though not exclusively) upon the two concluding chapters, about curves and surfaces, where the generalities are systematically applied. Hence it may be best to examine these later chapters first.

Curves (62 pp.). The Riemann-Roch theorem, Hurwitz's formula and Clifford's theorem are proved via sheaf cohomology. The canonical linear system is examined carefully, accompanied by a brief survey of moduli of curves, with examples. (Exercises include explicit deformations for curves of low genus, and Hurwitz's bound for the number of automorphisms when finite.) Elliptic curves are classified by j-invariant, using the Riemann-Roch theorem. The group structure is treated geometrically first, then using the bijection with Pic^0. Over $\mathbf{C}$, there is a good sketch of the connection with elliptic functions, including (with proof) a characterization of the curves with complex multiplication, in terms of the lattice structure. In characteristic p, the Hasse invariant is defined via Frobenius on $H^1(O)$, and the curves with Hasse = 0 are classified. There follows a short aside on rationality questions. The chapter concludes with a rigorous classification of all smooth complete curves of degree ≤ 7 in $\mathbf{P}^3$. Main theorems here are those of Halphen (a curve in $\mathbf{P}^n$ of genus $g \geq 2$ has a non-special very ample divisor of degree d if and only if $d \geq g+3$) and Castelnuovo (a curve of degree d in $\mathbf{P}^3$, not contained in any plane, has $d \geq 3$, and, for the genus g, we have $g \leq \frac{1}{4}d^2 - d + 1$ (d even), $g \leq \frac{1}{4}(d^2 - 1) - d + 1$ (d odd), with equality if and only if the curve lies on a quadric).

Surfaces (67 pp.). In the first 10 pages we find basic facts about linear equivalence and intersections of curves on a smooth projective surface, and then the following results: the Riemann-Roch theorem in its classical form, the adjunction formula, the Hodge index theorem, and the Nakai-Moišezon criterion for ample curves. (As corollaries, the inequality of Castelnuovo-Severi and its application to the Riemann hypothesis for curves are given as exercises!) There follows a thorough study of ruled surfaces, viewed as projective bundles $\mathbf{P}(E)$, for E locally free of rank 2 on a smooth curve. Rational and elliptic ruled surfaces classified via normal forms for E, and criteria for ample and very ample curves are deduced. A section on monoidal transformations culminates in a proof of embedded resolution for curves on a smooth projective surface. Then the nonsingular cubic surfaces in $\mathbf{P}^3$ are treated via the system of plane cubics through 6 general points; symmetries of the 27 lines yield a criterion for ample and very ample curves. There follows another 10-page section of generalities, this time on birational maps: factorization, invariance of p_a, Castelnuovo's criterion for contractible curves, existence of relatively minimal models. A survey of Enriques's classification, with references, concludes the exposition. (This chapter, far more than that on curves, requires the full power of schemes and cohomology: semicontinuity and base change, for example, unlock the ruled surfaces, while the theorem on formal functions verifies, in Castelnuovo's criterion, that the contracted surface is smooth.)

The two middle chapters on schemes (140 pp.) and cohomology (91 pp.) give a very good introduction to the essential ideas. Care has been taken to avoid setting up excessively elaborate machinery, without (the reviewer feels) unduly weakening the important theorems. Duality for projective varieties is neatly streamlined; coherence of direct images, Zariski's theorem on formal functions, and the semicontinuity and base-change theorems are given for projective, rather than proper, morphisms. The resulting exposition, although less general than that of EGA and its satellites, is much shorter, while still sufficient for the author's purposes. The abstract development itself is interwoven with some applications, for example to proving several versions of Bertini's theorem.

The text concludes with three appendices, sketching further developments (intersection theory and the Grothendieck Riemann-Roch theorem, algebraic varieties versus complex manifolds, the Weil conjectures) with references to the literature.

A course along the lines of this text, according to the introduction, ran for five quarters at Berkeley. Knowledge of basic commutative and homological algebra (or a willingness to learn it) is assumed; some familiarity with complex analysis might also help. A further necessary commitment is that of working and (as needed) discussing a sizable portion of the book's 464 exercises. (These exercises include important theorems, additional examples, alternate treatments of some topics, as well as historical and technical asides. The style of the exposition seems to draw the reader into the problems, so the experience of reading this book may be more active than is usual at this level.)

Granting these necessary commitments, the present text succeeds admirably, in the reviewer's opinion, in introducing its difficult subject at a level appropriate for preparing future workers in the field.

From MathSciNet, April 2007

Robert Speiser

MR1730819 (2001d:14002) 14A15 (14-01 14C05 14H50 14N05)

Eisenbud, David; Harris, Joe [Harris, Joseph Daniel]

The geometry of schemes. (English summary)

Graduate Texts in Mathematics, 197.

Springer-Verlag, *New York*, 2000. x+294 *pp*. \$69.95; \$26.95 paperbound.

ISBN 0-387-98638-3; 0-387-98637-5

This book is a wonderful introduction to the way of looking at algebraic geometry introduced by Alexandre Grothendieck and his school. The style of this book, however, differs greatly from that of Bourbaki; it is not formal and systematic, but friendly and inviting, like the style of David Mumford, whom both authors credit as being their teacher for the topics covered by this book. Thus this book introduces big ideas with seemingly simple, concrete examples, generalizes from them to an appropriate abstract formulation, and then applies the concept to interesting classical problems in a meaningful way. It is a pleasure to read.

The authors assume the reader is familiar with classical algebraic geometry and with some of the ideas of several complex manifolds. They also assume a competence in commutative algebra. Although they seem to start at the beginning in the study of schemes, apparently they expect the beginning reader not to attempt to fill in every detail, but to appreciate the general flavor of the subject and to go to other sources for details or help. Especially important resources in this regard are [D. Eisenbud, *Commutative algebra*, Springer, New York, 1995; MR1322960 (97a:13001)] and [R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977; MR0463157 (57 #3116)]. This book is informal, breezy, and refreshing. Many of the grubby details are left to the reader in the form of exercises or just left out. The classical ideas behind the definitions and constructions are brought in and used throughout. Notably missing from a complete introduction to schemes is any mention of cohomology, although finite free resolutions are used on occasion.

There are six chapters to the book. The first gives the definition of affine schemes with a brief, appropriate interlude on sheaf theory. Then schemes are pieced together from affine schemes. Similarities to and differences from differentiable manifolds are emphasized. Grothendieck's basic construction of fibered product is treated at length, although the details of the proof are not dwelt on; these are left to the reader as exercises. Also, the functor of points is introduced and used here to make sense of, among other things, the "points" of the fibered product.

The second chapter begins with the study of reduced affine schemes over an algebraically closed field; such schemes are essentially classical affine varieties. Then the authors relax the condition on the base field, and finally remove the reduced condition too. In each case they emphasize the advantages of considering the more general situations. They present, for example, the flat family of three lines through the origin in 3-space consisting of the union of the x-axis, the y-axis, and the pencil of lines through the origin in the plane defined by $x = y$. The limiting scheme as the pencil goes to the line in the xy-plane is the three lines with an embedded point at the origin. They thus make the use of schemes seem worth the effort. The book is liberally interspersed with such useful, concrete examples. They also emphasize the usefulness, and not just in number theory, of schemes over the ring of integers $\mathbf{Z}$.

The third chapter treats projective schemes. The authors work from the premise that having understood affine schemes, there is not so much more that is new; the differences between projective schemes and projective varieties are like those already treated between affine schemes and affine varieties. They start with the construction of Proj of a graded ring rather than with $\mathbf{P}^n$, although they discuss the advantages and disadvantages of both approaches. They discuss thoroughly the characterization of morphisms into projective n-space $\mathbf{P}^n$ in terms of 1-quotients of $\mathcal{O}_P^{n+1}$. They construct the Grassmannian in two ways, both locally and using Plücker coordinates. They use Hilbert's original approach to show that the Hilbert function $H(\nu)$ coincides with a polynomial for large ν. They prove Bezout's theorem for complete intersections of hypersurfaces in $\mathbf{P}^n$ over a field using the Koszul complex. This leads to more general formulations of intersection theory, and, eventually, after some examples, to a reference to W. Fulton [*Intersection theory*, Springer, Berlin, 1984; MR0732620 (85k:14004)].

The fourth chapter discusses the classical notions of the flexes of plane curves and blow-ups in great detail. The authors define flexes for curves even with multiple components, all the while emphasizing with examples what their results mean. They define blow-ups inefficiently, but with motivation. They begin by blowing up a point in the plane. Then they construct the blow-up as the closure of a graph when the base is affine. Next they characterize the blow-up by its universal property. Finally they construct the blow-up in the general case of a closed subscheme in an arbitrary scheme as Proj of a Rees algebra. Once again they illustrate with many examples. They end the chapter with a discussion of Fano schemes of linear subspaces in projective varieties, especially lines on quadric and cubic surfaces, and a discussion of forms.

The fifth chapter explains several attempts to define the image of a morphism, why one cannot do so, and what mathematical gems have come from the attempts, including resultants and discriminants. The sixth and final chapter returns to the functor of points. The authors describe the notion of a Zariski sheaf, and use it to show that many constructions in algebraic geometry turn out to be better described as representations of such sheaves. As an example they discuss the Hilbert polynomial and the Hilbert scheme at length, with some examples to show how these concepts help answer some old questions and introduce new ones in a natural way. They describe the tangent space to a Zariski sheaf, and use the dimension of this tangent space to bound the dimension of the corresponding component of the Hilbert scheme. They end with a mention of moduli spaces and algebraic stacks.

Every algebraic geometer will want this book. While the level of difficulty is uneven, the book is full of insights and useful tidbits. The authors have succeeded in their goals of showing schemes at work in a number of important areas of classical algebraic geometry, and showing how the language of schemes can be used to resolve problems that were awkward and even intractable in the older, more limited language. On the other hand, they retain the vital spirit of classical algebraic geometry alive for modern workers in the field.

From MathSciNet, April 2007

Allen B. Altman