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Differential geometry and analysis on CR manifolds, by Sorin Dragomir and Giuseppe Tomassini, Progress in Mathematics, vol. 246, Birkhäuser, Basel, 2006, xiv+487 pp., US$109.00, ISBN 978-0-8176-4388-1

Of the three mathematical terms in the title of the book under review, only *CR Manifold* is not part of standard training for mathematicians. We define this term below. We mention here that the etymology of the term is amusing, and CR stands both for Cauchy-Riemann and for complex-real.

The notion of CR manifold arose as the natural generalization of real submanifold in a complex manifold. The influential paper [6] of Chern and Moser in 1974 was entitled "Real hypersurfaces in complex manifolds". A real hypersurface is of course a real submanifold of real codimension one. The unit sphere in complex Euclidean space $\mathbf{C}^n$ and other real hypersurfaces arise naturally as the boundaries of domains. In 1956 Lewy [15, 16] discovered his famous example of an unsolvable PDE by investigating (what we now call) the CR geometry of the unit sphere.

One hundred years ago Levi observed that boundaries of domains of holomorphy satisfy a geometric property now known as pseudoconvexity. For smooth boundaries this condition can be expressed as the semi-definiteness of the Levi form, a complex variables analogue of the second fundamental form in Riemannian geometry. By forty-five years ago, motivated by Lewy's example and by the $\overline{\partial}$-Neumann problem, the subjects of *partial differential equations* and *complex analysis in several variables* began to interact intensively. The resulting symbiosis led to such notions as CR manifolds, CR functions, the tangential Cauchy-Riemann equations, local solvability, subellipticity, microlocal analysis, and so on.

The authors of the book under review focus on differential-geometric aspects of CR geometry. They observe in the preface, "While the analysis and PDE aspects seem to have captured most of the interest within the mathematical community, there has been, over the last ten or fifteen years, some effort to understand the differential-geometric aspect of the subject as well." Their bibliography of 449 references indicates that the words "some effort" qualify as understatement for effect.

Complex Euclidean space $\mathbf{C}^n$ is the vector space of n-tuples of complex numbers $(z_1, ..., z_n)$ with the standard Euclidean topology. It is natural to express complex vector fields as combinations of the $\frac{\partial}{\partial z_j}$ and the $\frac{\partial}{\partial \overline{z}_j}$ rather than in terms of the underlying real coordinate differentiations. A complex vector field is called of type $(1, 0)$ if it is a combination (with smooth coefficients) of the differentiations $\frac{\partial}{\partial z_j}$. Thus the complexified tangent bundle $T\mathbf{C}^n \otimes \mathbf{C}$ contains a subbundle $T^{1,0}\mathbf{C}^n$ whose sections are of type $(1, 0)$. Note that this subbundle is *integrable* in the sense that the Lie bracket of two $(1, 0)$ vector fields is also a $(1, 0)$ vector field. The complex conjugate bundle of $T^{1,0}\mathbf{C}^n$ is written $T^{0,1}\mathbf{C}^n$. Observe that $T^{1,0}\mathbf{C}^n \cap T^{0,1}\mathbf{C}^n = \{0\}$, and hence we can write

$$T\mathbf{C}^n \otimes \mathbf{C} = T^{1,0}\mathbf{C}^n \oplus T^{0,1}\mathbf{C}^n. \tag{1}$$

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When M is a real hypersurface in $\mathbf{C}^n$, its tangent spaces inherit some of the above structure. Let $TM \otimes \mathbf{C}$ denote the complexified tangent bundle. We define the bundle of $(1,0)$ vectors tangent to M by

$$T^{1,0}M = T^{1,0}\mathbf{C}^n \cap (TM \otimes \mathbf{C}), \tag{2}$$

and as above we write $T^{0,1}M$ for the complex conjugate bundle. Observe again that $T^{1,0}M \cap T^{0,1}M = \{0\}$. The fibre dimensions of $T^{1,0}M$ and $T^{1,0}M$ over $\mathbf{C}$ are each $n-1$, whereas the fibre dimension of $TM \otimes \mathbf{C}$ over $\mathbf{C}$ is $2n-1$. Therefore (1) fails when $\mathbf{C}^n$ is replaced by M, and there must be a missing direction. The missing direction is sometimes called the *bad direction*; it behaves differently from the other tangential directions. The anisotropic behavior of $TM \otimes \mathbf{C}$ plays a major role throughout CR geometry and has led to developments in partial differential equations, sub-Riemannian geometry, harmonic analysis, and geometric function theory.

The Heisenberg group H provides an important basic example of a CR manifold. We may realize H as a real hypersurface of $\mathbf{C}^n$, the boundary of the Siegel domain:

$$H = \{z \in \mathbf{C}^n : \mathrm{Im}(z_n) = \sum_{j=1}^{n-1} |z_j|^2\}. \tag{3}$$

Under a Cayley transformation the Siegel domain is biholomorphically equivalent to the unit ball. We may think of the unit sphere as the Heisenberg group together with a point at infinity. The same idea arises in elementary complex analysis. The unit disk in $\mathbf{C}$ is conformally equivalent to the upper half-plane, and the circle corresponds to the real line together with a point at infinity.

Let us consider the CR geometry of H. The $(1,0)$ vector fields L_j given by

$$L_j = \frac{\partial}{\partial z_j} + 2i\overline{z}_j \frac{\partial}{\partial z_n}$$

span the set of sections of $T^{1.0}(H)$ at each point, and their conjugates do the same for $T^{0,1}(H)$. The missing direction is given by

$$T = \frac{1}{2i}\left(\frac{\partial}{\partial z_n} + \frac{\partial}{\partial \overline{z}_n}\right).$$

Furthermore, we have the following commutation relations:

$$\begin{aligned} [L_j, L_k] &= 0, \\ [T, L_j] &= [T, \overline{L}_j] = 0, \\ [L_j, \overline{L}_k] &= \delta_{jk} T. \end{aligned} \tag{4}$$

The missing direction can be recovered by taking Lie brackets. At the Lie algebra level we see an example of a nilpotent Lie algebra of step two. Strongly pseudoconvex hypersurfaces osculate H to second order, and therefore harmonic analysis on H leads to general results for such hypersurfaces [10]. The anisotropic behavior on the tangent spaces distinguishes the situation in several variables from the case in one variable, where the bad direction is the only tangential direction. The interplay between the *good directions* and the bad direction leads to difficulties but also to exciting developments in many fields.

We now turn to the general setting of CR manifolds.

Definition. A CR manifold M is a smooth real manifold M together with the following structure on its tangent spaces. There is an integrable sub-bundle $T^{1,0}M$ of $TM \otimes \mathbf{C}$, with complex conjugate bundle $T^{0,1}M$, such that

$$T^{1,0}M \cap T^{0,1}M = \{0\}.$$

Suppose M has dimension m and $T^{1,0}M$ has complex rank n. The integer n is called the *CR dimension* of M, and the integer $m-2n$ is called the *CR codimension* of M. Thus there are n directions of type $(1,0)$, n other directions of type $(0,1)$, and $m-2n$ additional tangential directions. The CR manifold M is called of *hypersurface type* if its CR codimension is 1. Large parts of the text under review and of the literature in CR geometry concern CR manifolds of hypersurface type.

We next define the Levi form on an orientable CR manifold M of hypersurface type. The Levi form measures the extent to which the bundle $T^{1,0} \oplus T^{0,1}$ fails to be integrable. We first choose a globally defined nonvanishing real one-form θ that annihilates $T^{1,0} \oplus T^{0,1}$. The authors refer to θ as a pseudo-Hermitian structure on M. When the Levi form is nondegenerate, θ is a contact form, as $\theta \wedge (d\theta)^{n-1}$ is a volume form on M.

Definition. The Levi form λ_θ on M is the Hermitian form defined for local sections L and K of $T^{1,0}M$ by

$$\lambda_\theta(L, \overline{K}) = -id\theta(L, \overline{K}).$$

Using the Cartan formula for exterior derivative, one can express $\lambda_\theta(L, \overline{K})$ as the projection of the commutator $[L, \overline{K}]$ into the missing direction.

Formula (4) shows that the Levi form for the Heisenberg group corresponds to the identity matrix. By contrast the Levi form on a real hyperplane is identically zero. A CR manifold of hypersurface type is called *strongly pseudoconvex* if its Levi form is definite at each point. A CR manifold of hypersurface type is called *nondegenerate* if its Levi form is nondegenerate at each point, and in this case $T^{1,0} \oplus T^{0,1}$ is as far as possible from being integrable. The reviewer's book [7] deals primarily with methods for describing the local geometry of hypersurfaces at points where the Levi form degenerates. Those ideas play no role in the book under review, and we mention here only that in such situations techniques of algebraic geometry replace those of differential geometry.

Some general remarks about CR manifolds and their CR codimension help clarify the discussion. Observe that the definition of CR manifold makes sense even if we do not assume $T^{1,0}M$ to be integrable. In that case M is called an *almost CR manifold*. An almost CR manifold whose CR codimension is 0 is of course also known as an *almost complex manifold*. The Newlander-Nirenberg Theorem [17] states that an almost complex manifold for which $T^{1,0}M$ is integrable can be given the structure of a complex manifold.

Let M be a CR manifold; thus $T^{1,0}M$ is assumed to be integrable. When the CR codimension is 0, M has the structure of a complex manifold. We may think of complex manifolds as examples of CR manifolds and holomorphic mappings as examples of CR mappings (defined below), but the primary interest is when the CR codimension is positive. When the CR codimension is 1, M behaves like a real hypersurface in $\mathbf{C}^n$, but not all CR manifolds of hypersurface type can be realized (even locally) as embedded submanifolds of $\mathbf{C}^n$. At the other extreme, for CR codimension we have *totally real* n-dimensional submanifolds of $\mathbf{C}^n$; the rank of $T^{1,0}M$ is 0 and the CR codimension is n.

Andreotti and Hill [2] proved that real-analytic CR manifolds are locally embeddable. The embeddability problem is difficult in the smooth category. Kuranishi proved, using Nash-Moser techniques, that a strongly pseudoconvex CR manifold of dimension $2n-1$, where $2n-1 \geq 9$, can be locally embedded in $\mathbf{C}^{n+1}$. Akahori [1], using a Banach space implicit function theorem, and Webster [20], using integral formulas for solving $\overline{\partial}$, each extended the result to CR manifolds of dimension 7. An example of Nirenberg shows that CR manifolds of dimension 3 cannot in general be embedded. The problem is open in dimension 5. For manifolds of dimension at least 7, Catlin [5] proved a local embedding theorem under weaker assumptions on the Levi form. Many authors (see for example [J]) have contributed to various aspects of the embeddability problem for abstract CR manifolds. Such ideas are not the main focus of the book under review, but the authors nonetheless mention them and provide some references.

Corresponding to CR manifolds we have also the notions of CR mappings and CR functions. Let M and N be CR manifolds. A smooth mapping $f : M \to N$ is called a CR map if its derivative maps $T^{1,0}M$ to $T^{1,0}N$. Thus, for each point $p \in M$, we have

$$df(p)(T_p^{1,0}M) \subset T_{f(p)}^{1,0}N. \tag{5}$$

One has the obvious notion of CR equivalence; f is a CR equivalence if it is a CR mapping and also a diffeomorphism. Fefferman [8] proved in 1974 (a result since generalized and simplified by many authors) that a biholomorphic mapping between smoothly bounded strongly pseudoconvex domains extends to be a CR equivalence of the boundaries.

Of considerable importance are CR maps $f : M \to \mathbf{C}$; one calls them *CR functions*. A smooth function is a CR function if and only if it satisfies the tangential Cauchy-Riemann equations $\overline{L}(f) = 0$ for all sections $\overline{L}$ of $T^{0,1}M$. One important part of the theory not discussed in the book under review concerns regularity and extension results for CR functions. For example we could define CR functions by assuming $\overline{L}(f) = 0$ in the distribution sense and ask whether f must in fact be smooth. The restriction of a holomorphic function to a real submanifold M must be a CR function; conversely not every CR function can be extended to be holomorphic. There is a large literature on the extension of CR maps and related ideas, including papers going back to the 1930's. See for example the monographs [3] and [4].

The book under review emphasizes the differential geometry of CR manifolds, where metrics, connections, and curvature tensors dominate the discussion. The main concern is CR manifolds of hypersurface type with nondegenerate Levi form. When the Levi form is nondegenerate one can use it to define a semi-Riemannian metric on $T(M)$ called the Webster metric. Tanaka [18] and Webster [19] independently constructed a linear connection ∇, compatible with the Levi form, which plays a major role in this book. The Chern curvature tensor C can be computed from ∇, and C vanishes if and only if M is locally CR equivalent to the standard unit sphere. In this sense the unit sphere is the CR analogue of flat real Euclidean space.

Many topics in this book center around a Lorentz metric called the Fefferman metric. Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$. In conjunction with work on the boundary behavior of the Bergman kernel function and the Dirichlet problem for a complex Monge-Ampère equation, Fefferman [9] introduced a Lorentz metric on $S^1 \times b\Omega$. The conformal class of the Fefferman metric is invariant under biholomorphic mappings. Lee [12] extended its definition to an intrinsically defined circle bundle over a CR manifold of hypersurface type. To do so he used the pseudo-Hermitian structure form θ and the Tanaka-Webster connection ∇. Lee's construction leads to the CR Yamabe problem.

Yamabe first posed what became known as the *Yamabe problem* for Riemannian manifolds in 1960. Work by Yamabe, Trudinger, Aubin, and Schoen led to a complete solution by 1984. One is given a Riemannian manifold M with metric g. The aim is to multiply g by a conformal factor u such that the new metric has constant scalar curvature. The factor u then must satisfy a nonlinear PDE called the Yamabe equation. See the *Bulletin* Survey article [14] for more information.

The book under review sketches the situation in the Riemannian case and then turns to the CR Yamabe problem. The setting is a compact strictly pseudoconvex CR manifold of hypersurface type with pseudo-Hermitian structure one-form θ. The given metric is the Fefferman metric F_θ with pseudo-Hermitian scalar curvature ρ. We want to find a conformal factor which makes the new pseudo-Hermitian scalar curvature constant. We replace the contact form θ by $u^{\frac{2}{n}}\theta$. The Fefferman metric also transforms by a factor of $u^{\frac{2}{n}}$. The new contact form $u^{\frac{2}{n}}\theta$ has constant pseudo-Hermitian scalar curvature η if and only if u satisfies the CR Yamabe equation

$$c_n \Delta_b u + \rho u = \eta u^{\frac{n}{2}+1}, \tag{6}$$

where Δ_b is the (subelliptic) boundary sub-Laplacian and c_n is a constant. Equation (6) is the Euler-Lagrange equation for a variational problem involving the CR structure. One of the difficulties is of course establishing that an appropriate infimum is attained by some positive u. Jerison and Lee [12] solved the CR Yamabe problem for nonlocally spherical CR manifolds of dimension greater than 3; their work is presented in 53 pages in the book under review. The book mentions (without including any proofs) additional work by Gamara-Yacoub that completes the solution of the CR Yamabe problem. The book includes proofs of some but not all of the needed results of Folland-Stein [10] on harmonic analysis on the Heisenberg group.

Fefferman suggested nearly thirty years ago that the Bergman kernel on strongly pseudoconvex domains plays a role analogous to that of the heat kernel in Riemannian geometry. Many authors have contributed to this program by relating singularities of the Bergman kernel to the CR-invariant theory of Tanaka and Chern-Moser. Although the Fefferman metric is a major part of the development of the book under review, the book includes almost no information about this important aspect of Fefferman's exciting program.

Prefixes such as *pseudo-* and *quasi-*, and to a lesser extent words such as *almost*, plague complex analysis and CR geometry. The fundamental concept of pseudoconvexity for example is a complex analogue of real Euclidean convexity; it is similar enough to warrant a similar name but different enough to warrant a prefix such as *pseudo* to warn the reader of the differences. They matter! For example, *every* domain in **C** is pseudoconvex. Mathematicians with a good ear sometimes find names whose sound appeals to all. I wonder how a vote between "CR Geometry" and "Almost Pseudo-Hermitian Geometry" would turn out.

The book under review has chapters on pseudoharmonic maps, pseudo-Einsteinian manifolds, pseudo-Hermitian immersions, and quasiconformal mappings. In each case the authors do a good job of indicating the relationship to the situation without the warning prefix and also including most of the needed details in the situation with the prefix. In particular the chapter on pseudo-Hermitian immersions contains considerable nice material. The authors generally get the details right, and these chapters are worthwhile. Some readers will wish for fewer topics but with more discussion and synthesis of the ideas and computations. Because researchers from diverse parts of mathematics have contributed to CR geometry, different names and notations are used for the same concepts. The book under review makes an effort to merge the different chapters into a coherent whole, but some readers will wish for greater interweaving of the chapters.

Our discussion has shown CR geometry to be a vast subject requiring methods from partial differential equations, several complex variables, harmonic analysis, differential geometry, and so on. We have barely mentioned the connections with algebraic geometry, representation theory, and other fields. No one book can possibly cover all these topics. Readers of the book under review will be treated to a thorough treatment of the *differential geometric* aspects of CR geometry.

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John P. D'Angelo
University of Illinois, Urbana
E-mail address: jpda@math.uiuc.edu