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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
IGOR DOLGACHEV

MR0072877 (17,345d) 20.0X
Chevalley, Claude
Invariants of finite groups generated by reflections.
Amer. J. Math. **77** (1955), 778–782.

Soit G un group fini de transformations linéaires d'un espace vectoriel V sur un corps K de caractéristique 0. On suppose G engendré par des réflexions (involutions dont les éléments invariants forment un hyperplan). Présant le théorème classique des invariants pour les groupes finis, l'auteur montre que l'algèbre J sur K des invariants de G est engendré par l'unité et par un nombre d'invariants homogènes, algébriquement indépendants, égal à la dimension n de V. En outre, si $I_1, \cdots, I_n$ est un tel système de générateurs, m_i $(1 \leq i \leq n)$ le degré de I_i, le produit $\prod_{i=1}^n (1 - t^{m_i})$ est égal au produit de $(1-t)^n$ par un polynôme où le coefficient de t^k est la dimension des éléments homogènes de degre k dans l'algèbre S/F, S étant l'algèbre des polynômes sur V et F l'idéal homogène de S engendré par les invariants (polynômes) de G. Il en résulte que le produit $m_1 m_2 \cdots m_n$ est égal à l'ordre du groupe G; enfin l'auteur prouve que la représentation linéaire naturelle de G dans l'espace vectoriel S/F est équivalente à la représentation régulière de G.

From MathSciNet, October 2007

J. Dieudonné

MR0106428 (21 #5160) 50.00 (10.00)
Steinberg, Robert
Finite reflection groups.
Trans. Amer. Math. Soc. **91** (1959), 493–504.

The author considers, in Euclidean n-space, a finite set of hyperplanes which is symmetrical by reflection in each one. The reflecting hyperplanes decompose the space into congruent angular regions called chambers, any one of which will serve as a fundamental region for the group G generated by all the reflections. A sufficient set of generators consists of the reflections R_i in the n walls W_i of the fundamental region $(i = 1, 2, \cdots, n)$. The group G is said to be irreducible if the walls do not fall into two sets, all those in the first set being orthogonal to all those in the second. The products of the n reflections taken in various orders are all conjugate [Coxeter, Ann. of Math. (2) **35** (1934), 588–621; p. 602]; the period of such a product is denoted by h. It is possible to name the n walls in such an order that, for some s, the first s of them are mutually orthogonal, and likewise the remaining $n - s$. This clever trick enables the author to give general proofs for several theorems which had previously been observed by the reviewer and verified laboriously by separate consideration of individual cases. For instance, the total number of reflecting hyperplanes is $nh/2$; and if G contains the central inversion I then h is even and $I = (R_1 R_2 \cdots R_n)^{h/2}$ [Coxeter, loc. cit., pp. 606, 610].

Letting R_1W_2 denote the image of W_2 by reflection in W_1, and making the conventions $W_k = W_j$ and $R_k = R_j$ if $k \equiv j \pmod n$, the author expresses the $nh/2$ reflecting hyperplanes in the form

$$R_1R_2 \cdots R_{k-1}W_k \quad (k = 1, 2, \cdots, nh/2).$$

He finds that the reflecting hyperplanes, in this order, contain sets of $n-1$ consecutive vertices of a certain skew nh-gon, namely a "modified Petrie polygon" [Coxeter, *Regular polytopes*, Methuen, London, 1948; MR0027148 (10,261e); pp. 228–231].

The group G is said to be crystallographic if all the dihedral angles between adjacent walls are multiples of either $\pi/4$ or $\pi/6$. In this case it is possible to choose nh vectors $\pm\rho$, orthogonal to the reflecting hyperplanes, of such lengths that the adjunction of the translations 2ρ to G yields an infinite discrete group. Because of their application to the theory of simple Lie groups, these vectors are called roots, and those along inward normals to the n walls W_i, say α_i, are called fundamental roots (Freudenthal's "primary roots"). Every root is expressible in the form $\pm\sum x^i\alpha_i$, where the coefficients x^i are positive integers or zero. In particular, there is a so-called dominant root $\mu = \sum y^i\alpha_i$ whose coefficients are maximal. These vectors α_i and μ are $\frac{1}{2}\mathbf{t}_i$ and $\frac{1}{2}\mathbf{z}$ in the notation of the reviewer's *Extreme forms* [Canad. J. Math. **3** (1951), 391–441; MR0044580 (13,443c); pp. 404, 410]. The coefficients of α_i in the expression for the dominant root are related to h (and thence to the number of roots) by the simple formula $\sum y^i = h - 1$ [*Regular polytopes*, p. 234; *Extreme forms*, p. 413].

From MathSciNet, October 2007

H. S. M. Coxeter

MR0437798 (55 #10720) 32C40 (14B05 20G20)

Brieskorn, E.

Singular elements of semi-simple algebraic groups.

Actes du Congrès International des Mathématiciens (*Nice,* 1970)*, Tome* 2, pp. 279–284, *Gauthier-Villars, Paris*, 1971.

The Kleinian groups are the finite subgroups $H \subset \mathrm{SL}(2, \mathbf{C})$ (or SU(2)); they arise from the rotation groups of the regular solids (i.e., from finite subgroups of SO(3)). Klein showed $\mathbf{C}^2/H$ is a hypersurface in $\mathbf{C}^3$ (there are three generating H-invariant polynomials in $\mathbf{C}[x, y]$, with one relation); and P. Duval found that resolving these hypersurface singularities gives configurations of rational curves whose weighted dual graph is the Dynkin diagram of the complex Lie group A_n, D_n, E_6, E_7 or E_8. These singularities are called the rational double points (or RDP's). So, the Kleinian groups somehow correspond to simple Lie algebras of type A_k, D_k or E_k (e.g., the binary icosahedral group corresponds to E_8).

The author's main result is that the versal deformation $\mathcal{V} \to S$ of an RDP $\mathbf{C}^2/H$ resolves simultaneously in a family after finite base change $S' \to S$, where $S' \to S$ is Galois, with group W = Weyl group of A_k, D_k or E_k. That is, there is a smooth family $\mathcal{X} \to S'$, factoring via $f\colon \mathcal{V} \times S' \to S'$, which resolves fibre-wise the singularities of f. This result, suggested by the author's earlier papers [Math. Ann. **178** (1968), 255–270; MR0233819 (38 #2140)], is here treated in terms of algebraic groups.

If G is a complex simple algebraic group, T a maximal torus, W = Weyl group, then there is a map $\pi\colon G \to T/W$ associating to $x \in G$ the conjugacy class of its semi-simple part. (If $G = \mathrm{SL}(n)$, π sends a matrix to its characteristic polynomial.) π is smooth exactly for regular x (i.e., $\dim Z_G(x) = \mathrm{rk}\, G$). Grothendieck proved that π has a simultaneous resolution

$$\begin{array}{ccc} Y & \to & G \\ \downarrow & & \downarrow \pi \\ T & \to & T/W, \end{array}$$

where W consists of pairs (x, B), where B is a Borel subgroup containing x. This suggests consideration of the subregular elements of G, i.e., x such that $\dim Z_G(x) = \mathrm{rk}\, G + 2$ (the next lowest dimension). Results of Tits and Steinberg imply that the unipotent fibre U of π has a family of RDP singularities "along" the subregular locus V (these are the RDP's corresponding to G). Since U is locally a product along V, choose for $x \in V$ a smooth subvariety X of G transversal to V. Then the following theorem is announced. $(X, x) \to (T/W, e)$ is a versal deformation of the RDP. Simultaneous resolution after Galois base change now follows from Grothendieck's result.

The author mentions another result on the deformation space $S = T/W$ minus the discriminant locus; combining this with (later) work of Arnol′d and Deligne, that space is now known to be a $K(\pi, 1)$, where π is an extension of the Weyl group by a generalized braid group. [See, e.g., the author, Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pp. 21–44, Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973; MR0422674 (54 #10660).]

{Reviewer's remark: The proof of the author's main result is very briefly sketched in several sentences; many substantive details are missing, and a complete proof has never been written up. A number of people are now working to fill in the steps. However, the main corollary on simultaneous resolution of RDP's can be proved without this result on algebraic groups, using the author's earlier papers [cf. G. N. Tjurina, Funkcional. Anal. i Priložen. **4** (1970), no. 1, 77–83; MR0267129 (42 #2031)]. For A_n and D_n, the Weyl groups can be seen from the explicit construction. In general, one can use "elementary transformations" à la Burns-Rapoport or E. Horikawa to get a Weyl group action on the versal simultaneous resolution.}

{For the entire collection see MR0411875 (54 #4).}

From MathSciNet, October 2007

Jonathan M. Wahl

MR1066460 (92h:20002) 20-02 (20F32 20F55 20G15 20H15)

Humphreys, James E.

Reflection groups and Coxeter groups.

Cambridge Studies in Advanced Mathematics, 29.

Cambridge University Press, *Cambridge*, 1990. xii+204 *pp*. *ISBN* 0-521-37510-X

The symmetry groups of the regular polyhedra in $\mathbf{R}^n$ may be generated by reflections $s_1, \cdots, s_n$ and have a presentation with defining relations of the form $s_i^2 = (s_i s_j)^{k_{ij}} = 1$. In 1935, H. S. M. Coxeter enumerated all finite groups W generated by reflections and found that they are precisely the groups with presentations of this type, now called (finite) Coxeter groups. These groups and their infinite generalizations, some of which arise in elementary contexts as the symmetry groups of regular tesselations of Euclidean space, are the subject of this book. They have had extraordinary influence on geometry, Lie theory, finite groups, combinatorics and more distant parts of mathematics, for example singularity theory.

This is a useful book. The style is informal and the arguments are clear. The publisher describes it as a "graduate textbook" accessible to a reader with "a good knowledge of algebra" which "attempts to be both an introduction to Bourbaki and an updating of the coverage". This is fair billing. In its 200 pages it gives a readable introduction to Coxeter groups. It is the unique graduate level text on this subject and most of what it does is important for various aspects of Lie theory. The author keeps prerequisites to a minimum: the Euler characteristic of the Coxeter complex is computed without any topology and the section on invariants is written without any technicalities from commutative algebra or character theory. Occasional remarks hint at deeper connections with Lie theory. The book is more ambitious than the undergraduate text *Finite reflection groups* [second edition, Springer, New York, 1985; MR0777684 (85m:20001)] by L. C. Grove and C. T. Benson and is, of course, not as formidable as Bourbaki. It may attract browsers, but it does not convey the excitement of the geometry in Coxeter's *Regular polytopes* [Methuen, London, 1948; MR0027148 (10,261e)]. To do this would require, at a minimum, a leisurely historical introduction and more pictures to illustrate the remarkable amalgam of algebra and geometry in the theory of Coxeter groups; see, for example, the proof in Grove and Benson's book [op. cit.] that a finite reflection group has a Coxeter presentation.

Here are the chapter headings followed by parenthetical comments. (1) Finite reflection groups. (Bourbaki introduced root systems, assumed crystallographic, in Chapter 6, while Coxeter groups were introduced in Chapter 4. This goes against intuition. The author introduces the root system, without crystallographic restriction, at the start. Thus, by contrast with Bourbaki, one may introduce the set of positive roots made negative by an element of W and discuss parabolic subgroups and other basic facts with some reference to geometry. This chapter contains Steinberg's proof of the alternating sum formula for the Coxeter complex. Take care with the definition of inversion in Exercise 1.6.2.) (2) Classification of finite reflection groups. (Here are the classification of positive definite and semidefinite Coxeter graphs, a description of the irreducible root systems, and a brief description of the Coxeter groups of type E_6, E_7, E_8 with reference to the *Atlas of finite groups* [J. H. Conway et al., Oxford Univ. Press, Eynsham, 1985; MR0827219 (88g:20025)] concerning isomorphisms with classical groups over "small" fields.) (3) Polynomial invariants of finite reflection groups. (The exposition here is by now standard. The subtle two-dimensional argument concerning the action of a Coxeter element on the intersection of a plane with the fundamental domain is well written.) (4) Affine reflection groups. (Affine Weyl groups, the exchange condition, alcove and fundamental domain, a proof of the formula $|W| = n!c_1 \cdots c_n f$, where the c_i are the coefficients of the highest root of an irreducible group and f is the index of connection.) (5) Coxeter groups. (This is in part a selection of topics from Bourbaki's Chapters 4 and 5 beginning with the definition of Coxeter system and including the Tits cone, fundamental domain, and faithful canonical representation. Some topics not in Bourbaki include roots, following Deodhar, and the Bruhat order (with no attempt at algebro-geometric motivation, although a brief discussion of the facts in GL_3 would surely whet a beginner's appetite for this otherwise dry serving which is needed in Chapter 7).) (6) Special cases. (This covers finite Coxeter groups as finite reflection groups, crystallographic Coxeter groups, hyperbolic Coxeter groups and their classification by computer (three and one-half pages of diagrams) correcting the misprints in Bourbaki.) (7) Hecke algebras and Kazhdan-Lusztig polynomials. (The author writes that "the Hecke algebra is a sort of deformation of the group algebra of the related Weyl group....In any case, what we do is hard to motivate strictly in terms of Coxeter groups". True, but one page spent on Iwahori's theory for $\mathrm{GL}_2(\mathbf{F}_q)$ would have been better than nothing at all and three or so pages on $\mathrm{GL}_n(\mathbf{F}_q)$ would say a lot, without assuming any prerequisites. The treatment of the "generic algebra" following Couillens is surely an improvement on Bourbaki's infamous exercise. The aim of this chapter is proof of the existence and uniqueness of the Kazhdan-Lusztig polynomials. It does this quite clearly with full details in thirteen pages, but a beginner might not realize, from what is said here, that these polynomials are probably the most important development in the subject in the last 10–15 years.) (8) Complements. (This is a series of one-page introductions, without proofs, to various topics: the word problem; reflection subgroups; involutions; Coxeter elements and their eigenvalues; Möbius function of the Bruhat order: intervals and Bruhat graphs; shellability; automorphisms of the Bruhat order; Poincaré series of affine Weyl groups; representations of finite Coxeter groups; Schur multipliers; and (sound of trumpets) Coxeter groups and Lie theory.)

One might think that this highly structured, very well developed, and in a sense elementary subject has reached its peak and that there are few loose ends. Not so. Like most aspects of Lie theory it has its fountain of youth. New and extraordinary connections continue to be found with seemingly unrelated parts of mathematics. É. Cartan [Enseignment Math. **35** (1936), 177–200; Zbl **15**, 204] wrote: "C'est en quelque sorte une loi historique que les propriétés générales des groupes simples ont presque toutes été vérifiées d'abord sur les différents groupes et qu'on a ensuite cherché et trouvé une raison générale dispensant de l'examen des cas particuliers; je ne connais guère qu'une exception à cette loi." This "experimental" aspect of the subject explains a large part of its charm. Anyone can play.

{A list of errata and updated list of references is available from the author. A second printing is being prepared with corrections and small revisions.}

From MathSciNet, October 2007

Louis Solomon